

Degree-optimal Moving Frames for Rational Curves

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Abstract

We present an algorithm that, for a given vector \mathbf{a} of n relatively prime polynomials in one variable over an arbitrary field \mathbb{K} , outputs an $n \times n$ invertible matrix P with polynomial entries such that it forms a *degree-optimal moving frame* for the rational curve defined by \mathbf{a} . The first column of the matrix P consists of a minimal-degree Bézout vector (a minimal-degree solution to the univariate effective Nullstellensatz problem) of \mathbf{a} , and the last $n - 1$ columns comprise an optimal-degree basis, called a μ -basis, of the syzygy module of \mathbf{a} . To develop the algorithm, we prove several new theoretical results on the relationship between optimal moving frames, minimal-degree Bézout vectors, and μ -bases. In particular, we show how the degrees bounds of these objects are related. The theory developed in this paper allows us to compute an optimal moving frame by extension of a μ -basis algorithm presented earlier by Hong, Hough and Kogan. The main step of the algorithm is a partial row-echelon reduction of a $(2d + 1) \times (nd + n + 1)$ matrix over \mathbb{K} , where d is the maximum degree of the input \mathbf{a} . Our literature search did not yield any other algorithms for computing degree-optimal moving frames or minimal-degree Bézout vectors. We compare our algorithm with a non-optimal moving frame algorithm based on a generalized version of Euclid's extended gcd algorithm.

Keywords: rational curves, moving frames, effective univariate Nullstellensatz, Bézout identity and Bézout vectors, syzygies, μ -basis.

1 Introduction

Let $\mathbb{K}[s]$ denote a ring of polynomials over a field \mathbb{K} and let $\mathbb{K}[s]^n$ denote the set of row vectors of length n over \mathbb{K} . Let $GL_n(\mathbb{K}[s])$ denote the set of invertible $n \times n$ matrices over $\mathbb{K}[s]$, or equivalently, the set of matrices whose columns are *point-wise* linearly independent.

A nonzero vector $\mathbf{a} \in \mathbb{K}[s]^n$ defines a parametric curve in \mathbb{K}^n . A matrix $P \in GL_n(\mathbb{K}[s])$ can be viewed as assigning a basis of vectors in \mathbb{K}^n at each point of the curve.

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In other words, the columns of the matrix can be viewed as a coordinate system, or a frame, that moves along the curve. To be of interest, however, such assignment should not be arbitrary, but instead be related to the curve in a meaningful way. In this paper, we require that $\mathbf{a}P = [\gcd(\mathbf{a}), 0, \dots, 0]$, where $\gcd(\mathbf{a})$ is the monic greatest common divisor of the components of \mathbf{a} . We will call a matrix P with the above property a *moving frame at \mathbf{a}* . It is closely related to the notion of a “geometric moving frame” in differential geometry. A brief comparison with geometric moving frames is given in Section 6.2.

We observe that for any nonzero monic polynomial $\lambda(s)$, a moving frame at \mathbf{a} is also a moving frame at $\lambda\mathbf{a}$. Therefore, we can obtain an equivalent construction in the projective space $\mathbb{P}\mathbb{K}^{n-1}$ by considering only polynomial vectors \mathbf{a} such that $\gcd(\mathbf{a}) = 1$. Then P can be thought of as an element of $PGL_n(\mathbb{K}[s]) = GL_n(\mathbb{K}[s])/cI$, where $c \neq 0 \in \mathbb{K}$ and I is an identity matrix. A canonical map of \mathbf{a} to any of the affine subsets $\mathbb{K}^{n-1} \subset \mathbb{P}\mathbb{K}^{n-1}$ produces a rational curve in \mathbb{K}^n , and P assigns a projective moving frame at \mathbf{a} .

Obviously, the first column of a moving frame P at \mathbf{a} is a *Bézout vector* of \mathbf{a} ; that is, a vector comprised of the coefficients appearing in the output of the extended Euclidean algorithm. In Proposition 9, we prove that the last $n - 1$ columns of P comprise a point-wise linearly independent basis of the syzygy module of \mathbf{a} . The goal of this paper is to develop an efficient algorithm for constructing a *degree-optimal moving frame*, a moving frame that has column-wise minimal degree, as stated precisely in Definition 4. In Theorem 1, we show that a matrix P is a degree-optimal moving frame at \mathbf{a} if and only if the first column of P is a Bézout vector of \mathbf{a} of *minimal degree*, and the last $n - 1$ columns form a basis of the syzygy module of \mathbf{a} of *optimal degree*, called a μ -basis [3].

One may attempt to construct an optimal moving frame by putting together a minimal-degree Bézout vector and a μ -basis. This approach has the following disadvantages. First, our literature search did not yield any algorithm for computing a minimal-degree Bézout vector. Of course, one can compute such a vector by a brute-force method, namely by searching for a Bézout vector of a fixed degree, starting from degree zero, increasing the degree by one, and terminating the search once a Bézout vector is found, but this procedure is inefficient. Secondly, as the algorithm presented in this paper demonstrates, a minimal-degree Bézout vector and a μ -basis can be computed simultaneously which, of course, is a more efficient approach than computing them separately. Theorem 3, proved in this paper, is crucial for our algorithm, because it shows how a minimal-degree Bézout vector can be read off a Sylvester-type matrix associated with \mathbf{a} , the same matrix that has been used in [10] for computing a μ -basis. This theorem leads to an algorithm consisting of the following three steps: (1) build a Sylvester-type $(2d + 1) \times (nd + n)$ matrix A , associated with \mathbf{a} , where d is the maximal degree of the components of the vector \mathbf{a} , and append an additional column to A ; (2) run a single partial row-echelon reduction of the resulting $(2d + 1) \times (nd + n + 1)$ matrix; (3) read off an optimal moving frame from appropriate columns of the partial reduced row-echelon form.

Thus, the new results relating minimal-degree Bézout vectors, μ -bases, and optimal moving frames, established in this paper, allow us to obtain an algorithm for construct-

ing an optimal moving frame as an extension of the μ -basis algorithm presented in [10]. We implemented the algorithm in the computer algebra system Maple. The codes and examples are available on the web:

<http://www.math.ncsu.edu/~zchough/frame.html>

To make a comparison with previous algorithms, we note that algorithms for computing a μ -basis first appeared for the $n = 3$ case only in the works of Cox, Sederberg, and Chen [3], Zheng and Sederberg [16], and Chen and Wang [1]. The first algorithm for computing a μ -basis for polynomial vectors of arbitrary length n appeared in Song and Goldman [13] as a generalization of [1]. In [10], the first three authors of the current paper proposed an alternative algorithm for computing a μ -basis for arbitrary n , which exploits the periodicity of the non-pivotal column indices of a modified Sylvester-type matrix associated with \mathbf{a} . To our surprise, we did not find any published algorithm for computing a Bézout vector of minimal degree. We did, however, find an interesting algorithm for computing a *not-necessarily-optimal moving frame* in the book “Introduction to the Mathematical Theory of Systems and Control” by Polderman and Willems [12]. We discuss this algorithm and some other possible approaches in Section 6.1.

In addition to developing an algorithm for computing an optimal moving frame, we prove new results about the degrees of minimal-degree Bézout vectors and optimal moving frames. In Theorem 2 and Proposition 17, we establish relationships between the degrees of minimal-degree Bézout vectors and the vectors in a μ -basis. In particular, we prove that the minimal degree of a Bézout vector is bounded by the maximum of the degrees of vectors in a μ -basis. In Proposition 31 and Theorem 5, we establish sharp lower and upper bounds for the degree of an optimal moving frame and show that for a generic vector \mathbf{a} , the degree of an optimal moving frame equals to the sharp lower bound. The latter results are related to the μ -strata analysis performed in [2].

The paper is structured as follows. In Section 2, we give precise definitions of a degree-optimal moving frame, a minimal-degree Bézout vector, and a μ -basis, and we prove the essential properties of these objects. Theorem 1, proven in this section, states that a minimal-degree Bézout vector and a μ -basis are the building blocks of any degree-optimal moving frame. Additional new results of this section are stated in Theorem 2 and Propositions 9, 16, and 17, while Propositions 13 and 15 are adapted from [13].

In Section 3, by introducing a modified Sylvester-type matrix A , associated with an input vector \mathbf{a} , we reduce the problem of constructing a degree-optimal moving frame to a linear algebra problem over \mathbb{K} . Theorems 3 and 4 show how a minimal-degree Bézout vector and a μ -basis, respectively, can be constructed from the matrix A . Theorem 3 is new, while Theorem 4 is a slight modification of Theorem 27 in [10].

In Section 4, we prove new results about the degree bounds of an optimal moving frame. In particular, in Proposition 31, we establish the sharp lower bound $\left\lceil \frac{d}{n-1} \right\rceil$ and the sharp upper bound d for the degree of an optimal moving frame, and in Theorem 5, we prove that for a generic vector \mathbf{a} , the degree of every degree-optimal moving frame at \mathbf{a} equals to the sharp lower bound.

In Section 5, we present a degree-optimal moving frame (OMF) algorithm. The algorithm exploits the fact that the construction procedures for a minimal-degree Bézout vector and for a μ -basis, suggested by Theorems 3 and 4, can be accomplished simultaneously by a single partial row-echelon reduction of a $(2d+1) \times (nd+n+1)$ matrix over \mathbb{K} . In Proposition 36, we prove that the theoretical (worst-case asymptotic) complexity of the OMF algorithm equals to $O(d^2n + d^3 + n^2)$, and we trace the algorithm on our running example.

In Section 6, we discuss other ideas for constructing a moving frame; in particular, the ideas presented in [12]. We note, however, that none of the methods that we are aware of are guaranteed to lead to a degree-optimal moving frame. We also make a comparison with the differential geometry notion of a moving frame.

2 Moving frames, Bézout vectors, and syzygies

In this section, we give the definitions of moving frame and degree-optimal moving frame, make a precise formulation of the problem we consider, and explore the relationships between moving frames, syzygies, and Bézout vectors.

2.1 Basis definitions and notation

Throughout the paper, \mathbb{K} is an arbitrary field, $\overline{\mathbb{K}}$ is its algebraic closure, and $\mathbb{K}[s]$ is the ring of polynomials over \mathbb{K} . For arbitrary natural numbers t and m , by $\mathbb{K}[s]^{t \times m}$ we denote the set of $t \times m$ matrices with polynomial entries. The set of $n \times n$ invertible matrices over $\mathbb{K}[s]$ is denoted as $GL_n(\mathbb{K}[s])$. It is well-known and easy to show that the determinant of such matrices is a nonzero element of \mathbb{K} . For a matrix N , we will use notation N_{*i} to denote its i -th column. For a square matrix, $|N|$ denotes its determinant.

By $\mathbb{K}[s]^m$ we denote the set of vectors of length m with polynomial entries. All vectors are implicitly assumed to be *column vectors*, unless specifically stated otherwise. Superscript T denotes transposition. We will use the following definitions of the degree and leading vector of a polynomial vector:

Definition 1 (Degree and Leading Vector). *For $\mathbf{h} = [h_1, \dots, h_m] \in \mathbb{K}[s]^m$ we define the degree and the leading vector of \mathbf{h} as follows:*

- $\deg(\mathbf{h}) = \max_{i=1, \dots, m} \deg(h_i)$.
- $LV(\mathbf{h}) = [\text{coeff}(h_1, t), \dots, \text{coeff}(h_m, t)]^T \in \mathbb{K}^n$, where $t = \deg(\mathbf{h})$ and $\text{coeff}(h_i, t)$ denotes the coefficient of s^t in h_i .
- We will say that a set of polynomial vectors $\mathbf{h}_1, \dots, \mathbf{h}_k$ is degree-ordered if $\deg(\mathbf{h}_1) \leq \dots \leq \deg(\mathbf{h}_k)$

Example 2. Let $\mathbf{h} = \begin{bmatrix} 9 - 12s - s^2 \\ 8 + 15s \\ -7 - 5s + s^2 \end{bmatrix}$. Then $\deg(\mathbf{h}) = 2$ and $LV(\mathbf{h}) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

By $\mathbb{K}[s]_t^m$ we denote the set of vectors of length m of degree at most t .

2.2 Moving frame definition and problem statement

Definition 3 (Moving Frame). *For a given nonzero row vector $\mathbf{a} \in \mathbb{K}[s]^n$, with $n > 1$, a moving frame at \mathbf{a} is a matrix $P \in GL_n(\mathbb{K}[s])$, such that*

$$\mathbf{a}P = [\gcd(\mathbf{a}), 0, \dots, 0], \quad (1)$$

where $\gcd(\mathbf{a})$ denotes the greatest monic common divisor of \mathbf{a} .

For the case when \mathbb{K} is a finite field, we clarify that by a zero polynomial we mean a polynomial with all its coefficients equal to zero. As we will show below, a moving frame at \mathbf{a} always exists and is not unique. For instance, if P is a moving frame at \mathbf{a} , then a matrix obtained from P by permuting the last $n - 1$ columns of P is also a moving frame at \mathbf{a} . The set of all moving frames at \mathbf{a} will be denoted $\text{mf}(\mathbf{a})$. For more details on the structure of the set $\text{mf}(\mathbf{a})$, see Section 6.2. We are interested in constructing a moving frame of optimal degree.

Definition 4 (Degree-Optimal Moving Frame). *A moving frame P at \mathbf{a} is called degree-optimal if*

1. $\deg(P_{*2}) \leq \dots \leq \deg(P_{*n})$,
2. if P' is another moving frame at \mathbf{a} , such that $\deg(P'_{*2}) \leq \dots \leq \deg(P'_{*n})$, then
$$\deg(P_{*i}) \leq \deg(P'_{*i}) \quad \text{for } i = 1, \dots, n.$$

For simplicity, we will often use the term *optimal moving frame* instead of *degree-optimal moving frame*. A degree-optimal moving frame also is not unique, but it is clear from the definition that all optimal moving frames at \mathbf{a} have the same column-wise degrees. The main contribution of this paper is an algorithm for constructing a degree-optimal moving frame for an input polynomial vector \mathbf{a} with $\gcd(\mathbf{a}) = 1$. In other words, we solve the following:

Problem:

Input: $\mathbf{a} \neq 0 \in \mathbb{K}[s]^n$, row vector, such that $n > 1$, \mathbb{K} is a computable field,¹ and $\gcd(\mathbf{a}) = 1$.

Output: $P \in GL_n(\mathbb{K}[s])$, such that P is a degree-optimal moving frame at \mathbf{a} .

Example 5 (Running Example). *We will be using the following simple example throughout the paper to illustrate the theoretical ideas/findings and the resulting algorithm.*

Input: $\mathbf{a} = \begin{bmatrix} 2 + s + s^4 & 3 + s^2 + s^4 & 6 + 2s^3 + s^4 \end{bmatrix} \in \mathbb{Q}[s]^3$

Output: $P = \begin{bmatrix} 2 - s & 3 - 3s - s^2 & 9 - 12s - s^2 \\ 1 + 2s & 2 + 5s + s^2 & 8 + 15s \\ -1 - s & -2 - 2s & -7 - 5s + s^2 \end{bmatrix}$

One can immediately notice that the moving frame is closely related to the Bézout identity and to syzygies of \mathbf{a} . We explore and exploit this relationship in the following three subsections.

Throughout the paper, $\mathbf{a} \in \mathbb{K}[s]^n$ is assumed to be a nonzero row vector with $n > 1$.

¹A field is *computable* if there are algorithms for carrying out the arithmetic $(+, -, \times, /)$ operations among the field elements.

2.3 Bézout vectors

It is clear that the first column of a moving frame P at $\mathbf{a} = [a_1, \dots, a_n]$ consists of the components of the Bézout identity for \mathbf{a} .

$$P_{11} a_1 + \dots + P_{n1} a_n = \gcd(\mathbf{a}).$$

We use the term *Bézout vector* for the components of the Bézout identity:

Definition 6 (Bézout Vector). *A Bézout vector of a row vector $\mathbf{a} \in \mathbb{K}[s]^n$ is a column vector $\mathbf{h} = [h_1, \dots, h_n]^T \in \mathbb{K}[s]^n$, such that*

$$\mathbf{a} \mathbf{h} = \gcd(\mathbf{a}).$$

The set of all Bézout vectors of \mathbf{a} is denoted by $\text{Bez}(\mathbf{a})$ and the set of Bézout vectors of degree at most d is denoted $\text{Bez}_d(\mathbf{a})$.

Definition 7 (Minimal Bézout Vector). *A Bézout vector \mathbf{h} of $\mathbf{a} = [a_1, \dots, a_n] \in \mathbb{K}[s]^n$ is said to be of minimal degree if*

$$\deg(\mathbf{h}) = \min_{\mathbf{h}' \in \text{Bez}(\mathbf{a})} \deg(\mathbf{h}').$$

The existence of a Bézout vector can be proven using the extended Euclidean algorithm. Moreover, since the set of the degrees of all Bézout vectors is well-ordered, there is a minimal-degree Bézout vector. In this paper, we provide a simple linear algebra algorithm to construct a Bézout vector of minimal degree.

2.4 Syzygies and μ -bases

It is easy to see that the last $n - 1$ columns of a moving frame are syzygies:

Definition 8 (Syzygy). *A syzygy of a nonzero row vector $\mathbf{a} = [a_1, \dots, a_n] \in \mathbb{K}[s]^n$, for $n > 1$, is a column vector $\mathbf{h} \in \mathbb{K}[s]^n$, such that*

$$\mathbf{a} \mathbf{h} = 0.$$

The set of all syzygies of \mathbf{a} is denoted by $\text{syz}(\mathbf{a})$, and the set of syzygies of degree at most d is denoted $\text{syz}_d(\mathbf{a})$. It is easy to see that $\text{syz}(\mathbf{a})$ is a module. The next proposition shows that the last $n - 1$ columns of a moving frame form a basis of $\text{syz}(\mathbf{a})$.

Proposition 9 (Basis of Syzygies). *Let $P \in \text{mf}(\mathbf{a})$. Then the columns P_{*2}, \dots, P_{*n} form a basis of $\text{syz}(\mathbf{a})$.*

Proof. We need to show that P_{*2}, \dots, P_{*n} generate $\text{syz}(\mathbf{a})$ and that they are linearly independent over $\mathbb{K}[s]$.

1. From (1), it follows that $\mathbf{a} P_{*2} = \dots = \mathbf{a} P_{*n} = 0$. Therefore, $P_{*2}, \dots, P_{*n} \in \text{syz}(\mathbf{a})$. It remains to show that an arbitrary $\mathbf{h} \in \text{syz}(\mathbf{a})$ can be expressed as a linear combination of $P_{*2}, \dots, P_{*n} \in \text{syz}(\mathbf{a})$ over $\mathbb{K}[s]$. Trivially we have

$$\mathbf{h} = P(P^{-1}\mathbf{h}). \tag{2}$$

From (1), it follows that $\mathbf{a} = [\gcd(\mathbf{a}) \ 0 \ \cdots \ 0] P^{-1}$ and, therefore, the first row of P^{-1} is the vector $\tilde{\mathbf{a}} = \mathbf{a}/\gcd(\mathbf{a})$.

Hence, since $\mathbf{a}\mathbf{h} = 0$, then $P^{-1}\mathbf{h} = [0, g_2(s), \dots, g_n(s)]^T$ for some $g_i(s) \in \mathbb{K}[s]$, $i = 2, \dots, n$. Then (2) implies:

$$\mathbf{h} = \sum_{i=2}^n g_i P_{*i}.$$

Thus P_{*2}, \dots, P_{*n} generate $\text{syz}(\mathbf{a})$.

2. Let $f_2, \dots, f_n \in \mathbb{K}[s]$ be such that

$$f_2 P_{*2} + \cdots + f_n P_{*n} = 0. \quad (3)$$

We need to show that $f_2 = \cdots = f_n = 0$. Suppose otherwise. Then we can assume without loss of generality that $\gcd(f_2, \dots, f_n) = 1$ (otherwise just divide both sides of equation (3) by the gcd). Let s_0 be an arbitrary element of \mathbb{K} . Then there exists $i \in \{2, \dots, n\}$ such that $f_i(s_0) \neq 0$ (otherwise, we would have $(x - s_0) | f_2, \dots, (x - s_0) | f_n$, contradicting $\gcd(f_2, \dots, f_n) = 1$). By evaluating (3) on s_0 , we have

$$(f_2 P_{*2} + \cdots + f_n P_{*n})(s_0) = f_2(s_0) P_{*2}(s_0) + \cdots + f_n(s_0) P_{*n}(s_0) = 0.$$

Thus $P_{*2}(s_0), \dots, P_{*n}(s_0)$ are linearly dependent over \mathbb{K} and therefore $|P(s_0)| = 0$. Since

$$|P(s_0)| = |P|(s_0),$$

we have $|P|(s_0) = 0$. However, by the definition of a moving frame, $|P|$ is a nonzero constant. Contradiction implies that $f_2 = \cdots = f_n = 0$. Hence, P_{*2}, \dots, P_{*n} are linearly independent over $\mathbb{K}[s]$.

□

Remark 10. Note that the proof of Proposition 9 is valid over the ring of polynomials in several variables. It is well-known that in the multivariable case there exists \mathbf{a} for which $\text{syz}(\mathbf{a})$ is not free and then, from Proposition 9, it immediately follows that a moving frame at \mathbf{a} does not exist.

In the univariate case, it follows from the Hilbert Syzygy Theorem that $\text{syz}(\mathbf{a})$ is a free module of rank $n - 1$ for every nonzero vector \mathbf{a} . Moreover, in [3], Hilbert polynomials and the Hilbert Syzygy Theorem were used to show the existence of a basis of $\text{syz}(\mathbf{a})$ with especially nice properties, called a μ -basis. An alternative proof of the existence of a μ -basis based on elementary linear algebra was given in [10].

Definition 11 (μ -basis). For a nonzero row vector $\mathbf{a} \in \mathbb{K}[s]^n$, a set of $n - 1$ polynomial vectors $\mathbf{u}_1, \dots, \mathbf{u}_{n-1} \in \mathbb{K}[s]^n$ is called a μ -basis of \mathbf{a} , or, equivalently, a μ -basis of $\text{syz}(\mathbf{a})$, if the following two properties hold:

1. $LV(\mathbf{u}_1), \dots, LV(\mathbf{u}_{n-1})$ are linearly independent over \mathbb{K} ;

2. $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ generate $\text{syz}(\mathbf{a})$, the syzygy module of \mathbf{a} .

A μ -basis is, indeed, a *basis* of $\text{syz}(\mathbf{a})$ as justified by the following:

Lemma 12. *Let polynomial vectors $\mathbf{h}_1, \dots, \mathbf{h}_l \in \mathbb{K}[s]^n$ be such that $LV(h_1), \dots, LV(h_l)$ are linearly independent over \mathbb{K} . Then $\mathbf{h}_1, \dots, \mathbf{h}_l$ are linearly independent over $\mathbb{K}[s]$.*

Proof. Assume that $\mathbf{h}_1, \dots, \mathbf{h}_l$ are linearly dependent over $\mathbb{K}[s]$, i.e. there exist polynomials $f_1, \dots, f_l \in \mathbb{K}[s]$, not all zero, such that

$$\sum_{i=1}^l f_i \mathbf{h}_i = 0. \quad (4)$$

Let $t = \max_{i=1, \dots, l} \deg(f_i \mathbf{h}_i)$ and let \mathcal{I} be the set of indices on which this maximum is achieved. Then (4) implies

$$\sum_{i \in \mathcal{I}} LC(f_i) LV(\mathbf{h}_i) = 0,$$

where $LC(f_i)$ is the leading coefficient of f_i and is nonzero for $i \in \mathcal{I}$. This identity contradicts our assumption that $LV(\mathbf{h}_1), \dots, LV(\mathbf{h}_l)$ are linearly independent over \mathbb{K} . \square

In Propositions 13 below, we prove the properties of μ -bases, which are equivalent to its definition. The proof is adapted from [13] and only the properties used in the current paper are listed. For a more comprehensive list of properties of a μ -basis see Theorems 1 and 2 in [13].

Proposition 13 (Equivalent properties). *Let $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ be a degree-ordered basis of $\text{syz}(\mathbf{a})$, i.e. $\deg(\mathbf{u}_1) \leq \dots \leq \deg(\mathbf{u}_{n-1})$. Then the following statements are equivalent:*

1. [independence of the leading vectors] $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ is a μ -basis.
2. [reduced representation] For every $\mathbf{h} \in \text{syz}(\mathbf{a})$, there exist $f_1, \dots, f_{n-1} \in \mathbb{K}[s]$ such that $\deg(f_i \mathbf{u}_i) \leq \deg(\mathbf{h})$ and

$$\mathbf{h} = \sum_{i=1}^{n-1} f_i \mathbf{u}_i. \quad (5)$$

3. [optimality of the degrees] If $\mathbf{h}_1, \dots, \mathbf{h}_{n-1}$ is another basis of $\text{syz}(\mathbf{a})$, such that $\deg(\mathbf{h}_1) \leq \dots \leq \deg(\mathbf{h}_{n-1})$, then $\deg(\mathbf{u}_i) \leq \deg(\mathbf{h}_i)$ for $i = 1, \dots, n-1$.

Proof.

- (1) \implies (2) Since $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ is a basis of $\text{syz}(\mathbf{a})$, then for every $\mathbf{h} \in \text{syz}(\mathbf{a})$ there exist $f_1, \dots, f_{n-1} \in \mathbb{K}[s]$ such that (5) holds. Let $t = \max_{i=1, \dots, l} (\deg(f_i \mathbf{u}_i))$ and let \mathcal{I} be the set of indices on which this maximum is achieved. If $t > \deg(\mathbf{h})$, the equation (5) implies that

$$\sum_{i \in \mathcal{I}} LC(f_i) LV(\mathbf{u}_i) = 0,$$

where $LC(f_i)$ is the leading coefficient of f_i and is nonzero for $i \in \mathcal{I}$. This identity contradicts our assumption that $LV(\mathbf{u}_1), \dots, LV(\mathbf{u}_{n-1})$ are linearly independent over \mathbb{K} . Thus $t \leq \deg(\mathbf{h})$ as desired.

- (2) \implies (3) Assume there exists a degree-ordered basis $\mathbf{h}_1, \dots, \mathbf{h}_{n-1}$ of $\text{syz}(\mathbf{a})$ and an integer $k \in \{1, \dots, n-1\}$ such that $\deg(\mathbf{h}_k) < \deg(\mathbf{u}_k)$. Then there exists a matrix $H \in \mathbb{K}[s]^{(n-1) \times (n-1)}$, invertible over $\mathbb{K}[s]$, such that

$$[\mathbf{h}_1, \dots, \mathbf{h}_{n-1}] = [\mathbf{u}_1, \dots, \mathbf{u}_{n-1}] H.$$

However, from property (2) it follows that the upper right $k \times (n-k)$ block has only zero entries. This implies that $|H| = 0$. Contradiction.

- (3) \implies (1) Assume that $LV(\mathbf{u}_1), \dots, LV(\mathbf{u}_{n-1})$ are dependent. Then there exist constants $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{K}$, not all zero, such that

$$\alpha_1 LV(\mathbf{u}_1) + \dots + \alpha_{n-1} LV(\mathbf{u}_{n-1}) = 0. \quad (6)$$

Let k be the largest index such that α_k is non zero. Since $LV(\mathbf{u}_1)$ is a nonzero vector, $k > 1$. Let $d_i = \deg(\mathbf{u}_i)$ and consider a syzygy

$$\mathbf{h} = \alpha_k \mathbf{u}_k - \sum_{i=1}^{k-1} \alpha_i s^{d_k - d_i} \mathbf{u}_i.$$

Since \mathbf{u}_k is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{h}$, the set

$$\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_{n-1}, \mathbf{h}\} \quad (7)$$

also is a basis of $\text{syz}(\mathbf{a})$. From (6) it follows that $\deg(\mathbf{h}) < \deg(\mathbf{u}_k)$. If $\deg(\mathbf{h}) < \deg(\mathbf{u}_1)$, then the generating set $\mathbf{h}, \mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_{n-1}$ of $\text{syz}(\mathbf{a})$ is degree-ordered. This contradicts our assumption that $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ is a degree-optimal basis. If $\deg(\mathbf{h}) \geq \deg(\mathbf{u}_1)$, let $i \in \{1, \dots, k-1\}$ be maximal such that $\deg(\mathbf{h}) \geq \deg(\mathbf{u}_i)$. Then the set $\mathbf{u}_1, \dots, \mathbf{u}_i, \mathbf{h}, \mathbf{u}_{i+1}, \dots, \mathbf{u}_{k-1}, \mathbf{u}_{k+1}, \dots, \mathbf{u}_{n-1}$ is degree-ordered. Since $\deg(\mathbf{u}_{i+1}) > \deg(\mathbf{h})$ we again have a contradiction with our assumption that $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ is a degree-optimal basis.

□

We proceed with proving point-wise linear independence of the vectors in a μ -basis. In Theorem 1 of [13], μ -bases of real polynomial vectors were considered, and point-wise independence of the vectors in a μ -basis was proven for every s in \mathbb{R} . This proof can be word-by-word adapted to μ -bases of polynomial vectors over \mathbb{K} to show point-wise independence of vectors in a μ -basis for every s in \mathbb{K} . To prove Theorem 1 of our paper, however, we need a slightly stronger result: point-wise independence of the vectors in a μ -basis for every s in $\overline{\mathbb{K}}$. To arrive at this result, we first prove the following lemma. In this lemma and the following proposition, we use $\text{syz}_{\mathbb{K}[s]}(\mathbf{a})$ to denote the syzygy module of \mathbf{a} over the polynomial ring $\mathbb{K}[s]$, and $\text{syz}_{\overline{\mathbb{K}}[s]}(\mathbf{a})$ to denote the syzygy module of \mathbf{a} over the polynomial ring $\overline{\mathbb{K}}[s]$. Elsewhere, we use a shorter notation $\text{syz}(\mathbf{a}) = \text{syz}_{\mathbb{K}[s]}(\mathbf{a})$.

Lemma 14. *Let $\mathbf{a} \in \mathbb{K}[s]^n$ be nonzero. If $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ is a μ -basis of $\text{syz}_{\mathbb{K}[s]}(\mathbf{a})$, then $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ is a μ -basis of $\text{syz}_{\overline{\mathbb{K}}[s]}(\mathbf{a})$.*

Proof. Since $LV(\mathbf{u}_1), \dots, LV(\mathbf{u}_{n-1})$ are independent over \mathbb{K} , they also are independent over $\overline{\mathbb{K}}$. Thus, it remains to show that $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ generate $\text{syz}_{\overline{\mathbb{K}}[s]}(\mathbf{a})$. For an arbitrary $\mathbf{h} = [h_1, \dots, h_n]^T \in \text{syz}_{\overline{\mathbb{K}}[s]}(\mathbf{a})$, consider the field extension \mathbb{H} of \mathbb{K} generated by all the coefficients of the polynomials h_1, \dots, h_n . Then \mathbb{H} is a finite algebraic extension of \mathbb{K} and, therefore, by one of the standard theorems of field theory (see, for example, the first two theorems in Section 41 of [14]), \mathbb{H} is a finite-dimensional vector space over \mathbb{K} . Let $\gamma_1, \dots, \gamma_r \in \mathbb{H} \subset \overline{\mathbb{K}}$ be a vector space basis of \mathbb{H} over \mathbb{K} . By expanding each of the coefficients in \mathbf{h} in this basis, we can write \mathbf{h} as

$$\mathbf{h} = \gamma_1 \mathbf{w}_1 + \dots + \gamma_r \mathbf{w}_r, \quad (8)$$

for some $\mathbf{w}_1, \dots, \mathbf{w}_r \in \mathbb{K}[s]^n$. Multiplying by \mathbf{a} on the left, we get

$$0 = \gamma_1 \mathbf{a} \mathbf{w}_1 + \dots + \gamma_r \mathbf{a} \mathbf{w}_r. \quad (9)$$

Assume there exists $i \in \{1, \dots, r\}$ such that $\mathbf{a} \mathbf{w}_i \neq 0$. Let $k = \deg(\mathbf{a} \mathbf{w}_i)$ and let $b_j \in \mathbb{K}$ be the coefficient of the monomial s^k in the polynomial $\mathbf{a} \mathbf{w}_j$ for $j = 1, \dots, r$. Then, from (9), we have

$$0 = \gamma_1 b_1 + \dots + \gamma_r b_r.$$

Since $b_i \neq 0$, this contradicts the assumption that $\gamma_1, \dots, \gamma_r$ is a vector space basis of \mathbb{H} over \mathbb{K} . Thus, it must be the case that

$$\mathbf{a} \mathbf{w}_i = 0 \text{ for all } i = 1, \dots, r$$

and, therefore, (8) implies that the module $\text{syz}_{\overline{\mathbb{K}}[s]}(\mathbf{a})$ is generated by $\text{syz}_{\mathbb{K}[s]}(\mathbf{a})$. Since $\text{syz}_{\mathbb{K}[s]}(\mathbf{a})$ is generated by $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$, this completes the proof. \square

Proposition 15 (Point-wise independence over $\overline{\mathbb{K}}$). *If $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ is a μ -basis of $\text{syz}_{\mathbb{K}[s]}(\mathbf{a})$, then for any value $s \in \overline{\mathbb{K}}$, the vectors $\mathbf{u}_1(s), \dots, \mathbf{u}_{n-1}(s)$ are linearly independent over $\overline{\mathbb{K}}$.*

Proof. Suppose there exists $s_0 \in \overline{\mathbb{K}}$ such that $\mathbf{u}_1(s_0), \dots, \mathbf{u}_{n-1}(s_0)$ are linearly dependent over $\overline{\mathbb{K}}$. Then there exist constants $\alpha_1, \dots, \alpha_{n-1} \in \overline{\mathbb{K}}$, not all zero, such that

$$\alpha_1 \mathbf{u}_1(s_0) + \dots + \alpha_{n-1} \mathbf{u}_{n-1}(s_0) = 0.$$

Let $i = \max\{j \mid \alpha_j \neq 0\}$ and let

$$\mathbf{h} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_i \mathbf{u}_i.$$

Then $\mathbf{h} \in \text{syz}_{\overline{\mathbb{K}}[s]}(\mathbf{a})$ and is not identically zero, but $\mathbf{h}(s_0) = 0$. It follows that $\gcd(\mathbf{h}) \neq 1$ in $\overline{\mathbb{K}}[s]$ and, therefore, $\tilde{\mathbf{h}} = \frac{1}{\gcd(\mathbf{h})} \mathbf{h}$ belongs to $\text{syz}_{\overline{\mathbb{K}}[s]}(\mathbf{a})$ and has degree strictly less than the degree of \mathbf{h} .

By Lemma 14, $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ is a μ -basis of $\text{syz}_{\overline{\mathbb{K}}[s]}(\mathbf{a})$ and, since

$$\mathbf{u}_i = \frac{1}{\alpha_i} \left(\gcd(\mathbf{h}) \tilde{\mathbf{h}} - \alpha_1 \mathbf{u}_1 - \dots - \alpha_{i-1} \mathbf{u}_{i-1} \right),$$

the set of syzygies

$$\{\mathbf{u}_1, \dots, \mathbf{u}_{i-1}, \mathbf{u}_{i+1}, \dots, \mathbf{u}_{n-1}, \tilde{\mathbf{h}}\}$$

is a basis of $\text{syz}_{\overline{\mathbb{K}}[s]}(\mathbf{a})$. Ordering it by degree and observing that $\deg(\tilde{\mathbf{h}}) < \deg(\mathbf{h}) = \deg(\mathbf{u}_i)$ leads to a contradiction with the degree optimality property of a μ -basis. \square

2.5 The building blocks of a degree-optimal moving frame

From the discussions of the last section, it does not come as unexpected that a Bézout vector and a set of point-wise independent syzygies can serve as building blocks for a moving frame.

Proposition 16 (Building blocks of a moving frame). *For a nonzero $\mathbf{a} \in \mathbb{K}[s]^n$, let $\mathbf{h}_1, \dots, \mathbf{h}_{n-1}$ be elements of $\text{syz}(\mathbf{a})$ such that, for every $s \in \overline{\mathbb{K}}$, vectors $\mathbf{h}_1(s), \dots, \mathbf{h}_{n-1}(s)$ are linearly independent over $\overline{\mathbb{K}}$, and let \mathbf{h}_0 be a Bézout vector of \mathbf{a} . Then the matrix*

$$P = [\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_{n-1}]$$

is a moving frame at \mathbf{a} .

Proof. Clearly $\mathbf{a}P = [\gcd(\mathbf{a}), 0, \dots, 0]$. Let $\tilde{\mathbf{a}} = \frac{1}{\gcd(\mathbf{a})} \mathbf{a}$, then

$$\tilde{\mathbf{a}}P = [1, 0, \dots, 0]. \quad (10)$$

Assume that the determinant of P does not equal to a nonzero constant. Then there exists $s_0 \in \overline{\mathbb{K}}$ such that

$$|\mathbf{h}_0(s_0), \mathbf{h}_1(s_0), \dots, \mathbf{h}_{n-1}(s_0)| = 0$$

and, therefore, there exist constants $\alpha_0, \dots, \alpha_n \in \overline{\mathbb{K}}$, not all zero, such that

$$\alpha_0 \mathbf{h}_0(s_0) + \alpha_1 \mathbf{h}_1(s_0) + \dots + \alpha_{n-1} \mathbf{h}_{n-1}(s_0) = 0.$$

Multiplying on the left by $\tilde{\mathbf{a}}(s_0)$ and using (10), we get $\alpha_0 = 0$. Then

$$\alpha_1 \mathbf{h}_1(s_0) + \dots + \alpha_{n-1} \mathbf{h}_{n-1}(s_0) = 0$$

for some set of constants $\alpha_1, \dots, \alpha_{n-1} \in \overline{\mathbb{K}}$, not all zero. But this contradicts our assumption that for every $s \in \overline{\mathbb{K}}$, vectors $\mathbf{h}_1(s), \dots, \mathbf{h}_{n-1}(s)$ are linearly independent over $\overline{\mathbb{K}}$. Thus, the determinant of P equals to a nonzero constant, and therefore P is a moving frame. \square

Theorem 1. *A matrix P is a degree-optimal moving frame at \mathbf{a} if and only if P_{*1} is a Bézout vector of \mathbf{a} of minimal degree and P_{*2}, \dots, P_{*n-1} is a μ -basis of \mathbf{a} .*

Proof.

(\implies) Let P be a degree-optimal moving frame at \mathbf{a} . From Definition 4, it immediately follows that P_{*1} is a Bézout vector of \mathbf{a} of minimal degree. From Proposition 9, it follows that P_{*2}, \dots, P_{*n} is a basis of $\text{syz}(\mathbf{a})$. Assume P_{*2}, \dots, P_{*n} is not of optimal degree and, therefore, there exists a degree-ordered basis $\mathbf{h}_1, \dots, \mathbf{h}_{n-1}$ of $\text{syz}(\mathbf{a})$ and an integer $k \in \{1, \dots, n-1\}$, such that $\deg(\mathbf{h}_k) < \deg(P_{*k+1})$. Let $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ be a μ -basis of \mathbf{a} . Then, by the degree optimality property of a μ -basis, $\deg(\mathbf{u}_k) \leq \deg(\mathbf{h}_k) < \deg(P_{*k+1})$. From Proposition 15, it follows that the vectors $\mathbf{u}_1(s), \dots, \mathbf{u}_{n-1}(s)$ are independent for all $s \in \overline{\mathbb{K}}$. By Proposition 16, the matrix $P' = [P_{*1}, \mathbf{u}_1, \dots, \mathbf{u}_{n-1}]$ is a moving frame at \mathbf{a} . On the other hand, $\deg(P'_{*k+1}) = \deg(u_k) < \deg(P_{*k+1})$. This contradicts our assumption that P is degree-optimal. Therefore, P_{*2}, \dots, P_{*n} is a basis of $\text{syz}(\mathbf{a})$ of optimal degree and, therefore, is a μ -basis.

(\impliedby) Assume P_{*1} is a Bézout vector of \mathbf{a} of minimal degree and P_{*2}, \dots, P_{*n-1} is a μ -basis of \mathbf{a} . Then Proposition 15 implies that the vectors $P_{*2}(s), \dots, P_{*n-1}(s)$ are independent for all $s \in \overline{\mathbb{K}}$ and so P is a moving frame due to Proposition 16. Assume there exists a moving frame P' and an integer $k \in \{1, \dots, n\}$, such that $\deg(P'_{*k}) < \deg(P_{*k})$. If $k = 1$, then we have a contradiction with the assumption that P_{*1} is a Bézout vector of minimal degree. If $k > 1$, we have a contradiction with the degree optimality property of a μ -basis. Thus P satisfies Definition 4 of a degree-optimal moving frame. \square

Theorem 1 implies the following three-step process for constructing a degree-optimal moving frame at \mathbf{a} .

1. Construct a Bézout vector \mathbf{b} of \mathbf{a} of minimal degree.
2. Construct a μ -basis $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ of \mathbf{a} .
3. Let $P = [\mathbf{b}, \mathbf{u}_1, \dots, \mathbf{u}_{n-1}]$.

The main contribution of this paper is an algorithm that *simultaneously constructs a Bézout vector of minimal degree and a μ -basis*. Moreover, we are not aware of any other algorithms that produce a Bézout vector of minimal degree.

The next theorem shows that the degree of a minimal-degree Bézout vector of \mathbf{a} is bounded by the maximal degree of a μ -basis of \mathbf{a} . This result is repeatedly used in the paper.

Theorem 2. *For any nonzero $\mathbf{a} \in \mathbb{K}[s]^n$, and for any minimal-degree Bézout vector \mathbf{b} and any μ -basis $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ of \mathbf{a} , we have*

- 1) *if $\deg(\mathbf{a}) = \deg(\gcd(\mathbf{a}))$, then $\deg(\mathbf{b}) = 0$ and $\deg(\mathbf{u}_i) = 0$ for $i = 1, \dots, n-1$.*
- 2) *otherwise $\deg(\mathbf{b}) < \max_j \{\deg(\mathbf{u}_j)\}$.*

Proof.

- 1) The condition $\deg(\mathbf{a}) = \deg(\gcd(\mathbf{a}))$ implies that $\mathbf{a} = \gcd(\mathbf{a})v$, where v is a constant non-zero vector. In this case, it is obvious how to construct \mathbf{b} and $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$, each with constant components.

- 2) In this case, $\deg(\mathbf{a}) > \deg(\gcd(\mathbf{a}))$. The coefficient of $\mathbf{a}\mathbf{b}$ for $s^{\deg(\mathbf{a})+\deg(\mathbf{b})}$ is $LV(\mathbf{a})LV(\mathbf{b})$. By definition of Bézout vector, $\mathbf{a}\mathbf{b} = \gcd(\mathbf{a})$. Therefore, by our assumption, $\deg(\mathbf{a}\mathbf{b}) < \deg(\mathbf{a})$. Thus $LV(\mathbf{a})LV(\mathbf{b}) = 0$ or, in other words, $LV(\mathbf{b}) \in LV(\mathbf{a})^\perp$. Let $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ be a μ -basis of \mathbf{a} . By a similar argument, since $\mathbf{a}\mathbf{u}_j = 0$, we have $LV(\mathbf{u}_j) \in LV(\mathbf{a})^\perp$ for $j = 1, \dots, n-1$. By definition of a μ -basis, $LV(\mathbf{u}_j)$ are linearly independent, and so they form a basis for $LV(\mathbf{a})^\perp$. Therefore, there exist constants $\alpha_1, \dots, \alpha_{n-1}$ such that

$$LV(\mathbf{b}) = \sum_{j=1}^{n-1} \alpha_j LV(\mathbf{u}_j).$$

Suppose that $\deg(\mathbf{b}) \geq \max_j \{\deg(\mathbf{u}_j)\}$. Then define

$$\tilde{\mathbf{b}} = \mathbf{b} - \sum_{j=1}^{n-1} \alpha_j \mathbf{u}_j s^{\deg(\mathbf{b}) - \deg(\mathbf{u}_j)}.$$

Then $\mathbf{a}\tilde{\mathbf{b}} = \gcd(\mathbf{a})$ and $\deg(\tilde{\mathbf{b}}) < \deg(\mathbf{b})$, a contradiction to the minimality of $\deg(\mathbf{b})$. Therefore, $\deg(\mathbf{b}) < \max_j \{\deg(\mathbf{u}_j)\}$. □

The degree-optimality property of a μ -basis, stated in Proposition 13, insures that, although a μ -basis of \mathbf{a} is not unique, the ordered list of the degrees of a μ -basis of \mathbf{a} is unique. This list is called the μ -type of \mathbf{a} . A detailed analysis of the μ -strata of a set of polynomial vectors is given in [2]. In the next proposition, we show that, except for the upper bound provided by $\mu_{n-1} - 1$, no other additional restrictions on the degree of the minimal Bézout vector are imposed by the μ -type.

Proposition 17. *Fix $n \geq 2$. For all ordered lists of nonnegative integers $\mu_1 \leq \dots \leq \mu_{n-1}$, with $\mu_{n-1} \neq 0$, and for all $j \in \{0, \dots, \mu_{n-1} - 1\}$, there exists $\mathbf{a} \in \mathbb{K}[s]^n$ such that $\gcd(\mathbf{a}) = 1$ and*

1. *for any μ -basis $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ of \mathbf{a} , we have $\deg(\mathbf{u}_i) = \mu_i$, $i = 1, \dots, n-1$.*
2. *for any minimal-degree Bézout vector \mathbf{b} of \mathbf{a} , we have $\deg(\mathbf{b}) = j$.*

Proof. In the case when $n = 2$, given a non-negative integer μ_1 and an integer $j \in \{0, \dots, \mu_1 - 1\}$, take $\mathbf{a} = [s^{\mu_1-j}, s^{\mu_1} + 1]$. Then, obviously $\gcd(\mathbf{a}) = 1$, vector $\mathbf{b} = [-s^j, 1]^T$ is a minimal-degree Bézout vector, and vector $\mathbf{u}_1 = [s^{\mu_1} + 1, -s^{\mu_1-j}]^T$ is the minimal-degree syzygy, which in this case comprises a μ -basis of \mathbf{a} . Thus \mathbf{a} has the required properties.

In the case when $n \geq 3$, for the set of integers $\mu_1, \dots, \mu_{n-1}, j$ described in the proposition, take

$$\mathbf{a} = [s^{\mu_{n-1}-j}, s^{\mu_{n-1}-j+\mu_1}, s^{\mu_{n-1}-j+\mu_1+\mu_2}, \dots, s^{\mu_{n-1}-j+\mu_1+\dots+\mu_{n-2}}, s^{\mu_1+\dots+\mu_{n-1}} + 1].$$

Observe that $\gcd(\mathbf{a}) = 1$, and consider the matrix

$$P = \begin{bmatrix} & s^{\mu_1} & & & 1 \\ & -1 & s^{\mu_2} & & \\ & & -1 & \ddots & \\ -s^j & & & \ddots & s^{\mu_{n-1}} \\ 1 & & & & -s^{\mu_{n-1}-j} \end{bmatrix}.$$

It is easy to see that $\mathbf{a}P = [1, 0, \dots, 0]$ and $|P| = \pm 1$, so P is a moving frame at \mathbf{a} according to Definition 3. Therefore, the first column of P , i.e vector $\mathbf{b} = P_{*1}$, is a Bézout vector of \mathbf{a} , while the remaining columns $\mathbf{u}_1 = P_{*2}, \dots, \mathbf{u}_{n-1} = P_{*n}$ comprise a basis for the syzygy module of \mathbf{a} according to Proposition 9. Clearly $\deg(\mathbf{b}) = j$, while $\deg \mathbf{u}_i = \mu_i$ for $i = 1, \dots, n-1$.

The leading vectors of $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ are linearly independent and, therefore, vectors $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ comprise a μ -basis of \mathbf{a} . To prove that \mathbf{b} is of minimal degree, suppose, for the sake of contradiction, that there exists a vector $\mathbf{f} = [f_1, \dots, f_n]^T \in \mathbb{K}[s]^n$ with $\deg(\mathbf{f}) < j$ such that

$$f_1(s) a_1(s) + \dots + f_n(s) a_n(s) = 1 \text{ for all } s. \quad (11)$$

We observe that, since $\mu_{n-1} > 0$ and $j < \mu_{n-1}$, then $a_i(0) = 0$ for $i = 1, \dots, n-1$ and $a_n(0) = 1$. Then, by substituting $s = 0$ in (11), we get $f_n(0) = 1$ and, therefore, $f_n(s)$ is not a zero polynomial. This implies that $\deg(f_n a_n) = \mu_1 + \dots + \mu_{n-1} + \deg(f_n)$. Therefore, in order for the Bézout identity (11) to hold, at least one of the remaining $f_i a_i$, $i = 1, \dots, n-1$, must contain a monomial of degree $\mu_1 + \dots + \mu_{n-1} + \deg(f_n)$ as well. However, we assumed that $\deg(f_i) < j$ for all i , which implies that $\deg(f_i a_i) < \mu_1 + \dots + \mu_{n-1}$ for $i = 1, \dots, n-1$. Contradiction.

We thus conclude that \mathbf{a} has the required properties. \square

3 Reduction to a linear algebra problem over \mathbb{K}

In this section, we show that for a vector $\mathbf{a} \in \mathbb{K}[s]_d^n$ such that $\gcd(\mathbf{a}) = 1$, a Bézout vector of \mathbf{a} of minimal degree and a μ -basis of \mathbf{a} can be obtained from linear relationships among certain columns of a $(2d+1) \times (nd+n+1)$ matrix over \mathbb{K} .

3.1 Sylvester-type matrix A and its properties

For a nonzero polynomial row vector

$$\mathbf{a} = \sum_{0 \leq i \leq d} [c_{i1}, \dots, c_{in}] s^i \quad (12)$$

of length n and degree d , we correspond a $\mathbb{K}^{(2d+1) \times n(d+1)}$ matrix

$$A = \begin{bmatrix} c_{01} & \cdots & c_{0n} & & & & & & & & \\ \vdots & \cdots & \vdots & c_{01} & \cdots & c_{0n} & & & & & \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \ddots & & & & \\ c_{d1} & \cdots & c_{dn} & \vdots & \cdots & \vdots & \ddots & c_{01} & \cdots & c_{0n} & \\ & & & c_{d1} & \cdots & c_{dn} & \ddots & \vdots & \cdots & \vdots & \\ & & & & & & \ddots & \vdots & \cdots & \vdots & \\ & & & & & & & \ddots & \vdots & \cdots & \vdots & \\ & & & & & & & & c_{d1} & \cdots & c_{dn} & \end{bmatrix} \quad (13)$$

with the blank spaces filled by zeros. In other words, matrix A is obtained by taking $d + 1$ copies of a $(d + 1) \times n$ block of the coefficients of polynomials in \mathbf{a} . The blocks are repeated horizontally from left to right, and each block is shifted down by one relative to the previous one. Matrix A is related to the *generalized resultant matrix* R , appearing on page 333 of [15]. Indeed, if one takes the top-left $\mathbb{K}^{2d \times nd}$ submatrix of A , transposes this submatrix, and then permutes certain rows, one obtains R . However, the size and shape of the matrix A turns out to be crucial to our construction.

Example 18. For the row vector \mathbf{a} in the running example (Example 5), we have $n = 3$, $d = 4$,

$$c_0 = [2, 3, 6], \quad c_1 = [1, 0, 0], \quad c_2 = [0, 1, 0], \quad c_3 = [0, 0, 2], \quad c_4 = [1, 1, 1]$$

and

$$A = \begin{bmatrix} 2 & 3 & 6 & & & & & & & & \\ 1 & 0 & 0 & 2 & 3 & 6 & & & & & \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 & 3 & 6 & & \\ 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 3 & 6 \\ 1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 3 & 6 \\ & & & 1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & \\ & & & & & & 1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\ & & & & & & & & & 1 & 1 & 1 & 0 & 0 & 2 \\ & & & & & & & & & & & & 1 & 1 & 1 \end{bmatrix}.$$

A visual periodicity of the matrix A is reflected in the periodicity property of its non-pivotal columns which we are going to precisely define and exploit below. We remind readers of the definition of pivotal and non-pivotal columns.

Definition 19. A column of any matrix N is called pivotal if it is either the first column and is nonzero or it is linearly independent of all previous columns. The rest of the columns of N are called non-pivotal. The index of a pivotal (non-pivotal) column is called a pivotal (non-pivotal) index.

From this definition, it follows that every non-pivotal column can be written as a linear combination of the preceding *pivotal columns*.

We denote the set of pivotal indices of A as p and the set of its non-pivotal indices as q . The following two lemmas, proved in [10] (Lemma 17, 19) show how the specific structure of the matrix A is reflected in the structure of the set of non-pivotal indices q .

Lemma 20 (periodicity). *If $j \in q$ then $j + kn \in q$ for $0 \leq k \leq \left\lfloor \frac{n(d+1)-j}{n} \right\rfloor$. Moreover,*

$$A_{*j} = \sum_{r < j} \alpha_r A_{*r} \implies A_{*j+kn} = \sum_{r < j} \alpha_r A_{*r+kn}, \quad (14)$$

where A_{*j} denotes the j -th column of A .

Definition 21. *Let q be the set of non-pivotal indices. Let $q/(n)$ denote the set of equivalence classes of q modulo n . Then the set $\tilde{q} = \{\min \varrho \mid \varrho \in q/(n)\}$ will be called the set of basic non-pivotal indices. The remaining indices in q will be called periodic non-pivotal indices.*

Example 22. *For the matrix A in Example 18, we have $n = 3$ and $q = \{8, 9, 11, 12, 14, 15\}$. Then $q/(n) = \{\{8, 11, 14\}, \{9, 12, 15\}\}$ and $\tilde{q} = \{8, 9\}$.*

Lemma 23. *There are exactly $n - 1$ basic non-pivotal indices: $|\tilde{q}| = n - 1$.*

3.2 Isomorphism between $\mathbb{K}[s]_t^m$ and $\mathbb{K}^{m(t+1)}$

The second ingredient that we use to reduce our problem to a linear algebra problem over \mathbb{K} is an explicit isomorphism between vector spaces $\mathbb{K}[s]_t^m$ and $\mathbb{K}^{m(t+1)}$. Any polynomial m -vector \mathbf{h} of degree at most t can be written as $\mathbf{h} = w_0 + sw_1 + \cdots + s^t w_t$ where $w_i = [w_{1i}, \dots, w_{mi}]^T \in \mathbb{K}^m$. It is clear that the map

$$\begin{aligned} \sharp_t^m: \mathbb{K}[s]_t^m &\rightarrow \mathbb{K}^{m(t+1)} \\ \mathbf{h} &\rightarrow \mathbf{h}^{\sharp_t^m} = \begin{bmatrix} w_0 \\ \vdots \\ w_t \end{bmatrix} \end{aligned} \quad (15)$$

is linear. It is easy to check that the inverse of this map

$$\flat_t^m: \mathbb{K}^{m(t+1)} \rightarrow \mathbb{K}[s]_t^m$$

is given by a linear map:

$$v \rightarrow v^{\flat_t^m} = S_t^m v \quad (16)$$

where

$$S_t^m = \begin{bmatrix} I_m & sI_m & \cdots & s^t I_m \end{bmatrix} \in \mathbb{K}[s]^{m \times m(t+1)}.$$

Here I_m denotes the $m \times m$ identity matrix. For the sake of notational simplicity, we will often write \sharp , \flat and S instead of \sharp_t^m , \flat_t^m and S_t^m when the values of m and t are clear from the context.

Example 24. *For $\mathbf{h} \in \mathbb{Q}_3^3[s]$ given by*

$$\mathbf{h} = \begin{bmatrix} 9 - 12s - s^2 \\ 8 + 15s \\ -7 - 5s + s^2 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ -7 \end{bmatrix} + s \begin{bmatrix} -12 \\ 15 \\ -5 \end{bmatrix} + s^2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

we have

$$\mathbf{h}^\sharp = [9, 8, -7, -12, 15, -5, -1, 0, 1]^T.$$

Note that

$$\mathbf{h} = (\mathbf{h}^\sharp)^\flat = S \mathbf{h}^\sharp = \begin{bmatrix} I_3 & sI_3 & s^2I_3 \end{bmatrix} \mathbf{h}^\sharp.$$

With respect to the isomorphisms \sharp and \flat , the \mathbb{K} -linear map $\mathbf{a}: \mathbb{K}[s]_d^n \rightarrow \mathbb{K}[s]_{2d}$ corresponds to the \mathbb{K} linear map $A: \mathbb{K}^{n(d+1)} \rightarrow \mathbb{K}^{2d+1}$ in the following sense:

Lemma 25. Let $\mathbf{a} = \sum_{0 \leq j \leq d} c_j s^j \in \mathbb{K}_d^n[s]$ and $A \in \mathbb{K}^{(2d+1) \times n(d+1)}$ defined as in (13).

Then for any $v \in \mathbb{K}^{n(d+1)}$ and any $\mathbf{h} \in \mathbb{K}[s]_d^n$:

$$\mathbf{a} v^{\flat_d^n} = (Av)^{\flat_{2d}^1} \text{ and } (\mathbf{a} \mathbf{h})^{\sharp_{2d}^1} = A \mathbf{h}^{\sharp_d^n}. \quad (17)$$

The proof of Lemma 25 is straightforward. The proof of the first equality is explicitly spelled out in [10] (see Lemma 10). The second equality follows from the first and the fact that \flat_t^m and \sharp_t^m are mutually inverse maps.

Example 26. Consider the row vector \mathbf{a} in the running example (Example 5) and its associated matrix A (Example 18). Let $v = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]^T$. Then

$$Av = [26, 60, 98, 143, 194, 57, 62, 63, 42]^T$$

and so

$$(Av)^{\flat_{2d}^1} = S_8^1(Av) = 26 + 60s + 98s^2 + 143s^3 + 194s^4 + 57s^5 + 62s^6 + 63s^7 + 42s^8.$$

On the other hand, since

$$v^{\flat_d^n} = S_4^3 v = \begin{bmatrix} 1 + 4s + 7s^2 + 10s^3 + 13s^4 \\ 2 + 5s + 8s^2 + 11s^3 + 14s^4 \\ 3 + 6s + 9s^2 + 12s^3 + 15s^4 \end{bmatrix},$$

we have

$$\begin{aligned} \mathbf{a} v^{\flat_d^n} &= \begin{bmatrix} 2 + s + s^4 & 3 + s^2 + s^4 & 6 + 2s^3 + s^4 \end{bmatrix} \begin{bmatrix} 1 + 4s + 7s^2 + 10s^3 + 13s^4 \\ 2 + 5s + 8s^2 + 11s^3 + 14s^4 \\ 3 + 6s + 9s^2 + 12s^3 + 15s^4 \end{bmatrix} \\ &= 42s^8 + 63s^7 + 62s^6 + 57s^5 + 194s^4 + 143s^3 + 98s^2 + 60s + 26. \end{aligned}$$

We observe that

$$\mathbf{a} v^{\flat_d^n} = (Av)^{\flat_{2d}^1}.$$

We now wish to prove a result about the rank of A . First, we need the following.

Lemma 27. For all $\mathbf{a} \in \mathbb{K}[s]^n$ with $\gcd(\mathbf{a}) = 1$ and $\deg(\mathbf{a}) = d$ and all $i = 0, \dots, 2d$, there exist vectors $\mathbf{h}_i \in \mathbb{K}[s]^n$ such that $\deg(\mathbf{h}_i) \leq d$ and $\mathbf{a} \mathbf{h}_i = s^i$.

Proof. Let $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ be a μ -basis of \mathbf{a} . We will proceed by induction on i . Induction basis: For $i = 0$, the statement follows immediately from Theorem 2 and the well-known fact that $\text{syz}(\mathbf{a})$ can be generated by vectors of degree at most d (see, for example, [10] or [13]).

Induction step: Assume the statement is true in the i -th case i.e. there exists $\mathbf{h}_i \in \mathbb{K}[s]^n$ with $\deg(\mathbf{h}_i) \leq d$ such that $\mathbf{a} \mathbf{h}_i = s^i$ ($i \leq 2d - 1$). Then $\mathbf{a}(s\mathbf{h}_i) = s^{i+1}$. Let $\tilde{\mathbf{h}} = s\mathbf{h}_i$. Since $\deg(\mathbf{h}_i) \leq d$, it follows that $\deg(\tilde{\mathbf{h}}) \leq d + 1$. If $\deg(\tilde{\mathbf{h}}) \leq d$, let $\mathbf{h}_{i+1} = \tilde{\mathbf{h}}$ and we are done. Otherwise, $\deg(\tilde{\mathbf{h}}) = d + 1$. Following a similar argument as in Theorem 2, the coefficient of $\tilde{\mathbf{a}}\tilde{\mathbf{h}}$ for s^{2d+1} is $LV(\mathbf{a})LV(\tilde{\mathbf{h}})$, and since we assumed $i \leq 2d - 1$, it must be that $LV(\mathbf{a})LV(\tilde{\mathbf{h}}) = 0$. Thus, there exist constants $\alpha_1, \dots, \alpha_{n-1}$ such that

$$LV(\tilde{\mathbf{h}}) = \sum_{j=1}^{n-1} \alpha_j LV(\mathbf{u}_j).$$

Then define

$$\mathbf{h}_{i+1} = \tilde{\mathbf{h}} - \sum_{j=1}^{n-1} \alpha_j \mathbf{u}_j s^{d+1-\deg(\mathbf{u}_j)}.$$

Then $\mathbf{a} \mathbf{h}_{i+1} = s^{i+1}$ and $\deg(\mathbf{h}_{i+1}) < \deg(\tilde{\mathbf{h}})$, which means $\deg(\mathbf{h}_{i+1}) \leq d$. \square

We now prove the following result about the rank of matrix A , which can be compared with the statements about the rank of a different Sylvester-type matrix, R , given in Section 2 of [15].

Lemma 28. *For a nonzero polynomial vector \mathbf{a} of degree d , defined by (12), such that $\gcd(\mathbf{a}) = 1$, the corresponding matrix A , defined by (13), has rank $2d + 1$.*

Proof. By Lemma 27, for all $i = 0, \dots, 2d$, there exist vectors $\mathbf{h}_i \in \mathbb{K}[s]^n$ with $\deg(\mathbf{h}_i) \leq d$ such that $\mathbf{a} \mathbf{h}_i = s^i$. Observe that $(s^i)^\sharp = e_{i+1}$. Since $(\mathbf{a} \mathbf{h}_i)^\sharp = A \mathbf{h}_i^\sharp$, it follows that there exist vectors $\mathbf{h}_i^\sharp \in \mathbb{K}^{n(d+1)}$ such that $A \mathbf{h}_i^\sharp = e_j$ for all $j = 1, \dots, 2d + 1$. This means the range of A is \mathbb{K}^{2d+1} and hence $\text{rank}(A) = 2d + 1$. \square

3.3 Bézout vector of minimal degree

In this section, we construct a Bézout vector of \mathbf{a} of minimal degree by finding an appropriate solution to the linear equation

$$A v = e_1, \text{ where } e_1 = [1, 0, \dots, 0]^T \in \mathbb{K}^{2d+1}. \quad (18)$$

The following lemma establishes a one-to-one correspondence between the set $\text{Bez}_d(\mathbf{a})$ of Bézout vectors of \mathbf{a} of degree at most d and the set of solutions to (18).

Lemma 29. *Let $\mathbf{a} \in \mathbb{K}[s]_d^n$ be a nonzero vector such that $\gcd(\mathbf{a}) = 1$. Then $\mathbf{b} \in \mathbb{K}[s]_d^n$ belongs to $\text{Bez}_d(\mathbf{a})$ if and only if \mathbf{b}^\sharp is a solution of (18). Also $v \in \mathbb{K}^{n(d+1)}$ solves (18) if and only if v^\flat belongs to $\text{Bez}_d(\mathbf{a})$.*

Proof. Follows immediately from (17) and the observation that $e_1^{b_{2d}^1} = 1$. \square

Thus, our goal is to construct v such that v solves (18) and v^\flat is a Bézout vector of \mathbf{a} of minimal degree. To accomplish this, we recall that, when $\gcd(\mathbf{a}) = 1$, Lemma 28 asserts that $\text{rank}(A) = 2d + 1$. Therefore, A has exactly $2d + 1$ pivotal indices, which we can list in the increasing order $p = \{p_1, \dots, p_{2d+1}\}$. The corresponding columns of matrix A form a basis of \mathbb{K}^{2d+1} and, therefore, $e_1 \in \mathbb{K}^{2d+1}$ can be expressed as a unique linear combination of the pivotal columns:

$$e_1 = \sum_{j=1}^{2d+1} \alpha_j A_{*p_j}. \quad (19)$$

Define vector $v \in \mathbb{K}^{2d+1}$ by setting its p_j -th element to be α_j and all other elements to be 0. We prove that $\mathbf{b} = v^\flat$ is a Bézout vector of \mathbf{a} of minimal degree.

Theorem 3 (Minimal-Degree Bézout Vector). *Let $\mathbf{a} \in \mathbb{K}[s]_d^n$ be a polynomial vector with $\gcd(\mathbf{a}) = 1$, and let A be the corresponding matrix defined by (13). Let $p = \{p_1, \dots, p_{2d+1}\}$ be the pivotal indices of A , and let $\alpha_1, \dots, \alpha_{2d+1} \in \mathbb{K}$ be defined by the unique expression (19) of the vector $e_1 \in \mathbb{K}^{2d+1}$ as a linear combination of the pivotal columns of A . Define vector $v \in \mathbb{K}^{2d+1}$ by setting its p_j -th element to be α_j for $j = 1, \dots, 2d + 1$ and all other elements to be 0, and let $\mathbf{b} = v^\flat$. Then*

1. $\mathbf{b} \in \text{Bez}_d(\mathbf{a})$
2. $\deg(\mathbf{b}) = \min_{\mathbf{b}' \in \text{Bez}(\mathbf{a})} \deg(\mathbf{b}')$.

Proof. 1. From (19), it follows immediately that $Av = e_1$. Therefore, by Lemma 29, we have that $\mathbf{b} = v^\flat \in \text{Bez}_d(\mathbf{a})$.

2. To show that \mathbf{b} is of minimal degree, we rewrite (19) as

$$e_1 = \sum_{j=1}^k \alpha_j A_{*p_j}, \quad (20)$$

where k is the largest integer between 1 and $2d + 1$, such that $\alpha_k \neq 0$. Then the last nonzero entry of v appears in p_k -th position and, therefore,

$$\deg(\mathbf{b}) = \deg(v^\flat) = \lceil p_k/n \rceil - 1. \quad (21)$$

Assume that $\mathbf{b}' \in \text{Bez}(\mathbf{a})$ is such that $\deg(\mathbf{b}') < \deg(\mathbf{b})$. Then $\mathbf{b}' \in \text{Bez}_d(\mathbf{a})$ and therefore $Av' = e_1$, for $v' = \mathbf{b}'^\sharp = [v'_1, \dots, v'_{n(d+1)}] \in \mathbb{K}^{n(d+1)}$. Then

$$e_1 = \sum_{j=1}^{n(d+1)} v'_j A_{*j} = \sum_{j=1}^r v'_j A_{*j}, \quad (22)$$

where r is the largest integer between 1 and $n(d + 1)$, such that $v'_r \neq 0$. Then

$$\deg(\mathbf{b}') = \lceil r/n \rceil - 1 \quad (23)$$

and since we assumed that $\deg(\mathbf{b}') < \deg(\mathbf{b})$, we conclude from (21) and (23) that $r < p_k$.

On the other hand, since all non-pivotal columns are linear combinations of the preceding pivotal columns, we can rewrite (22) as

$$e_1 = \sum_{j \in \{1, \dots, 2d \mid p_j \leq r < p_k\}} \alpha'_j A_{*p_j} = \sum_{j=1}^{k-1} \alpha'_j A_{*p_j}. \quad (24)$$

By the uniqueness of the representation of e_1 as a linear combination of the A_{*p_j} , the coefficients in the expansions (20) and (24) must be equal, but $\alpha_k \neq 0$ in (20). Contradiction. \square

In the algorithm presented in Section 5, we exploit the fact that the coefficients α 's in (20) needed to construct a minimal-degree Bézout vector of \mathbf{a} can be read off the reduced row echelon form $[\hat{A}|\hat{v}]$ of the augmented matrix $[A|e_1]$. On the other hand, as was shown in [10] and reviewed in the next section, the coefficients of a μ -basis of \mathbf{a} also can be read off the matrix \hat{A} . Therefore, a μ -basis is constructed as a byproduct of our algorithm for constructing a Bézout vector of minimal degree.

3.4 μ -bases

In [10], we showed that the coefficients of a μ -basis of \mathbf{a} can be read off the basic non-pivotal columns of matrix A (recall Definition 21). Recall that according to Lemma 23, the matrix A has exactly $n - 1$ basic non-pivotal columns.

Theorem 4 (μ -Basis). *Let $\mathbf{a} \in \mathbb{K}[s]_d^n$ be a polynomial vector, and let A be the corresponding matrix defined by (13). Let $\tilde{q} = [\tilde{q}_1, \dots, \tilde{q}_{n-1}]$ be the basic non-pivotal indices of A , ordered increasingly. For $i = 1, \dots, n - 1$, a basic non-pivotal column $A_{*\tilde{q}_i}$ is a linear combination of the previous pivotal columns:*

$$A_{*\tilde{q}_i} = \sum_{\{r \in p \mid r < \tilde{q}_i\}} \alpha_{ir} A_{*r}, \quad (25)$$

for some $\alpha_{ir} \in \mathbb{K}$. Define vector $b_i \in \mathbb{K}^{2d+1}$ by setting its \tilde{q}_i -th element to be 1, its r -th element to be $-\alpha_{ir}$ for $r \in p$ such that $p_j < \tilde{q}_i$, and all other elements to be 0. Then the set of polynomial vectors

$$\mathbf{u}_1 = b_1^b, \quad \dots, \quad \mathbf{u}_{n-1} = b_{n-1}^b$$

is a degree-ordered μ -basis of \mathbf{a} .

Proof. The fact that $\mathbf{u}_1 = b_1^b, \dots, \mathbf{u}_{n-1} = b_{n-1}^b$ is a μ -basis of \mathbf{a} is the statement of Theorem 27 of [10]. By construction, the last nonzero entry of vector b_i is in the \tilde{q}_i -th position, and therefore for $i = 1, \dots, n - 1$,

$$\deg(\mathbf{u}_i) = \deg(b_i^b) = \lceil \tilde{q}_i/n \rceil - 1.$$

Since the indices in \tilde{q} are ordered increasingly, the vectors $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ are degree-ordered. \square

The algorithm presented in Section 5 exploits the fact that the coefficients α 's in (25) are already computed in the process of computing a Bézout vector of \mathbf{a} .

4 Degree bounds for optimal moving frames

Similarly to the degree of a polynomial vector (Definition 1), we define the degree of a polynomial matrix to be the maximum of the degrees of its entries. Obviously, for a given vector \mathbf{a} , all degree-optimal moving frames have the same degree. In this section, we establish the sharp upper and lower bounds on the degree of optimal moving frames. We also show that, for generic inputs, the degree of an optimal moving frame equals to the lower bound.

An alternative simple proof of the bounds could be given using the fact that, when $\gcd(\mathbf{a}) = 1$, the sum of the degrees of a μ -basis of \mathbf{a} equals to $\deg(\mathbf{a})$ (see Theorem 2 in [13]), along with the result relating the degree of a minimal-degree Bézout vector and the maximal degree of a μ -basis in Theorem 2 of the current paper. For the sharpness of the lower bound and its generality, one could use Proposition 3.3 of [2], determining the dimension of the set of vectors of a given μ -type, again combined with Theorem 2 of the current paper. Our results on the upper bound differ from what can be deduced from [2], because we allow components of \mathbf{a} to be linearly dependent over \mathbb{K} . To keep the presentation self-contained, we give the proofs based on the results of the current paper. We will repeatedly use the following lemma.

Lemma 30. *Let $\mathbf{a} \in \mathbb{K}[s]^n$ be nonzero and let A be the corresponding matrix (13). Furthermore, let k be the maximum among the basic non-pivotal indices of A . Then the degree of any optimal moving frame at \mathbf{a} equals to $\lceil \frac{k}{n} \rceil - 1$.*

Proof. It is straightforward to check that the maximal degree of the μ -basis, constructed in Theorem 4, has degree $\lceil \frac{k}{n} \rceil - 1$. From the optimality of the degrees property in Proposition 13, it follows that for any two degree-ordered μ -bases $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ and $\mathbf{u}'_1, \dots, \mathbf{u}'_{n-1}$ of \mathbf{a} and for $i = 1, \dots, n-1$, we have $\deg(\mathbf{u}_i) = \deg(\mathbf{u}'_i)$. Therefore, the maximum degree of vectors in any μ -basis equals to $\lceil \frac{k}{n} \rceil - 1$. Theorem 2 implies that the degree of any optimal moving frame equals to the maximal degree of a μ -basis. \square

In the next proposition, we establish sharp lower and upper bounds for the degree of an optimal moving frame.

Proposition 31. *Let $\mathbf{a} \in \mathbb{K}[s]^n$ with $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$. Then for every degree-optimal moving frame P at \mathbf{a} , we have $\lceil \frac{d}{n-1} \rceil \leq \deg(P) \leq d$, and these degree bounds are sharp. By sharp, we mean that for all $n > 1$ and $d > 0$, there exists an $\mathbf{a} \in \mathbb{K}[s]^n$ with $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$ such that, for every degree-optimal moving frame P at \mathbf{a} , we have $\deg(P) = \lceil \frac{d}{n-1} \rceil$. Likewise, for all $n > 1$ and $d > 0$, there exists an $\mathbf{a} \in \mathbb{K}[s]^n$ with $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$ such that, for every degree-optimal moving frame P at \mathbf{a} , we have $\deg(P) = d$.*

Proof.

1. (lower bound): Let P be an arbitrary degree-optimal moving frame at \mathbf{a} . Then the following equation holds

$$\mathbf{a}P = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

It follows from Cramer's rule that

$$\mathbf{a}_i = \frac{(-1)^{i+1}}{|P|} |P_{i,1}| \quad i = 1, \dots, n,$$

where $P_{i,1}$ denotes the submatrix of P obtained by removing the 1-st column and the i -th row. We remind the reader that $|P|$ is a nonzero constant. Assume for the sake of contradiction that $\deg(P) < \left\lceil \frac{d}{n-1} \right\rceil$. Then $\deg(P) < \frac{d}{n-1}$. Since $|P_{i,1}|$ is the determinant of an $(n-1) \times (n-1)$ submatrix of P , we have $\deg(\mathbf{a}_i) = \deg(|P_{i,1}|) < (n-1) \frac{d}{n-1} = d$ for all $i = 1, \dots, n$. This contradicts the assumption that $\deg(\mathbf{a}) = d$. Thus, $\deg(P) \geq \left\lceil \frac{d}{n-1} \right\rceil$.

We will prove that the lower bound $\left\lceil \frac{d}{n-1} \right\rceil$ is sharp by showing that, for all $n > 1$ and $d > 0$, the following matrix

$$P = \left[\begin{array}{c|cccccc} 1 & & & & & -s^{d-k\lceil \frac{d}{n-1} \rceil} \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & -s^{\lceil \frac{d}{n-1} \rceil} & \\ & & & & 1 & -s^{\lceil \frac{d}{n-1} \rceil} \\ & & & & & \ddots & \ddots \\ & & & & & & \ddots & -s^{\lceil \frac{d}{n-1} \rceil} \\ & & & & & & & 1 \end{array} \right] \quad (26)$$

has degree $\left\lceil \frac{d}{n-1} \right\rceil$ and is a degree-optimal moving frame at the vector

$$\mathbf{a} = \left[1, 0, \dots, 0, s^{d-k\lceil \frac{d}{n-1} \rceil}, \dots, s^{d-2\lceil \frac{d}{n-1} \rceil}, s^{d-1\lceil \frac{d}{n-1} \rceil}, s^{d-0\lceil \frac{d}{n-1} \rceil} \right]. \quad (27)$$

Here $k \in \mathbb{N}$ is the maximal such that $d > k \left\lceil \frac{d}{n-1} \right\rceil$ (explicitly $k = \left\lceil \frac{d}{\lceil \frac{d}{n-1} \rceil} \right\rceil - 1$), the number of zeros in \mathbf{a} is $n - k - 2$, the upper-left block of P is of the size $(n - k - 1) \times (n - k - 1)$, the lower-right block is of the size $(k + 1) \times (k + 1)$, and the other two blocks are of the appropriate sizes.

First, we show that such \mathbf{a} and P actually exist (not just optically). That is, the number of zeros in \mathbf{a} is non-negative, and the upper-left block in P exists; in other words, $n - 1 \geq k + 1$. Suppose otherwise. Then we would have

$$d - k \left\lceil \frac{d}{n-1} \right\rceil \leq d - (n-1) \left\lceil \frac{d}{n-1} \right\rceil \leq 0$$

which contradicts the condition $d > k \left\lceil \frac{d}{n-1} \right\rceil$.

Second, P is a degree-optimal moving frame at \mathbf{a} . Namely,

- (a) $\mathbf{a}P = [1, 0, \dots, 0]$, so P is a moving frame at \mathbf{a} .
- (b) The first column of P , $[1, 0, \dots, 0]^T$, is a minimal-degree Bézout vector of \mathbf{a} .
- (c) The last $n - 1$ columns of P are syzygies of \mathbf{a} , and since $P \in \text{mf}(\mathbf{a})$, by Proposition 9, they form a basis of $\text{syz}(\mathbf{a})$. It is easy to see that these columns have linearly independent leading vectors as well. Thus, they form a μ -basis of \mathbf{a} .

Finally, we show that the degree of P is the lower bound, i.e. $\left\lceil \frac{d}{n-1} \right\rceil$. From inspection of the entries of P , we see immediately that

$$\deg(P) = \max \left\{ d - k \left\lceil \frac{d}{n-1} \right\rceil, \left\lceil \frac{d}{n-1} \right\rceil \right\}.$$

It remains to show that $d - k \left\lceil \frac{d}{n-1} \right\rceil \leq \left\lceil \frac{d}{n-1} \right\rceil$. Suppose not. Then

$$d > (k+1) \left\lceil \frac{d}{n-1} \right\rceil,$$

a contradiction to the maximality of k . Thus, $\deg(P) = \left\lceil \frac{d}{n-1} \right\rceil$. Hence, we have proved that the lower bound is sharp.

2. (upper bound): From Theorems 3 and 4, it follows immediately that d is an upper bound of a degree-optimal moving frames. We will prove that the upper bound d is sharp by showing that, for all $n > 1$ and $d > 0$, the following matrix of degree d

$$P = \begin{bmatrix} 1 & & & & -s^d \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}. \quad (28)$$

is the degree-optimal moving frame for the vector

$$\mathbf{a} = [1, 0, \dots, 0, s^d]$$

Indeed:

- (a) $\mathbf{a}P = [1, 0, \dots, 0]$ and so P is a moving frame at \mathbf{a} .
- (b) The first column of P , $[1, 0, \dots, 0]^T$, is a minimal-degree Bézout vector of \mathbf{a} .
- (c) The last $n - 1$ columns of P are syzygies of \mathbf{a} , and since $P \in \text{mf}(\mathbf{a})$, by Proposition 9, they form a basis of $\text{syz}(\mathbf{a})$. It is easy to see that these columns have linearly independent leading vectors as well. Thus, they form a μ -basis of \mathbf{a} .

□

In Theorem 5 below, we show that for generic $\mathbf{a} \in \mathbb{K}[s]^n$ with $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$, and for all degree-optimal moving frames P at \mathbf{a} , $\deg(P) = \left\lceil \frac{d}{n-1} \right\rceil$. To prove the theorem, we need the following lemmas, where we will use notation

$$k = \text{quo}(d, n-1) \text{ and } r = \text{rem}(d, n-1).$$

Lemma 32. *For arbitrary $\mathbf{a} \in \mathbb{K}[s]^n$ with $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$, the principal $d+k+1$ submatrix of the associated matrix A has the form*

$$C = \begin{bmatrix} c_{01} & \cdots & \cdots & c_{0n} & & & & & & & \\ \vdots & \cdots & \cdots & \vdots & c_{01} & \cdots & \cdots & c_{0n} & & & \\ \vdots & \cdots & \cdots & \vdots & \vdots & \cdots & \cdots & \vdots & \ddots & & \\ c_{d1} & \cdots & \cdots & c_{dn} & \vdots & \cdots & \cdots & \vdots & \ddots & c_{01} & \cdots & c_{0,r+1} \\ & & & & c_{d1} & \cdots & \cdots & c_{dn} & \ddots & \vdots & \cdots & \vdots \\ & & & & & & & & \ddots & \vdots & \cdots & \vdots \\ & & & & & & & & & c_{d1} & \cdots & c_{d,r+1} \end{bmatrix}, \quad (29)$$

where C consists of k full $(d+1) \times n$ size blocks and 1 partial block of size $(d+1) \times (r+1)$.

Proof. If we take k full $(d+1) \times n$ blocks and 1 partial $(d+1) \times (r+1)$ block, then the number of columns of C is $nk + r + 1 = (n-1)k + r + k + 1 = d + k + 1$, as desired. Furthermore, since the leftmost block takes up the first $d+1$ rows of C , and we shift the block down by 1 a total of k times, the number of rows of C is $d+k+1$ as well. □

Lemma 33. *Let $\mathbf{a} \in \mathbb{K}[s]^n$ with $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$, and let C be the principal $d+k+1$ submatrix of A given by (29). If C is nonsingular, then for any degree-optimal moving frame P at \mathbf{a} , we have $\deg(P) = \left\lceil \frac{d}{n-1} \right\rceil$.*

Proof. If C is nonsingular, then first $d+k+1$ columns of the matrix A are pivotal columns. Since $\text{rank}(A) = 2d+1$, there are $d-k$ additional pivotal columns in A and, from the structure of A , each of the last $d-k$ blocks of A contain exactly one of these additional pivotal columns. All other columns in A are non-pivotal. We now consider two cases:

- 1) If $n-1$ divides d , then $r = 0$ and $k = \frac{d}{n-1} = \left\lceil \frac{d}{n-1} \right\rceil$. Thus, there is one column in the partial block in C , and so the remaining $n-1$ columns in this $(k+1)$ -th block of A are basic non-pivotal columns. Since in total there are $n-1$ basis non-pivotal columns, the largest basic non-pivotal index equals to $n(k+1)$, and therefore by Lemma 30, the degree of any optimal moving frame at \mathbf{a} is $\left\lceil \frac{d}{n-1} \right\rceil$.
- 2) If $n-1$ does not divide d , then $r > 0$ and $k = \left\lfloor \frac{d}{n-1} \right\rfloor$. Thus, there are at least two columns in the partial block in C , and so there are at most $n-2$ basic non-pivotal columns in the $(k+1)$ -th block of A . Since there are a total of $n-1$

basis non-pivotal columns, and all but one of the columns in the $(k+2)$ -th block are non-pivotal, the largest basic non-pivotal column index will appear in the $(k+2)$ -th block. Therefore, this largest index equals to $n(k+1) + j$ for some $1 \leq j \leq n$. By Lemma 30, the degree of any optimal moving frame at \mathbf{a} equals to $\left\lceil \frac{n(k+1)+j}{n} \right\rceil - 1 = k + 1 = \left\lfloor \frac{d}{n-1} \right\rfloor + 1 = \left\lceil \frac{d}{n-1} \right\rceil$.

□

Lemma 34. *For all $n > 1$ and $d > 0$, there exists a vector $\mathbf{a} \in \mathbb{K}[s]^n$ with $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$ such that $\det(C) \neq 0$.*

Proof. Let $n > 1$ and $d > 0$. We will find a suitable witness for \mathbf{a} . Recalling the relation $d = k(n-1) + r$, we will consider the following three cases:

- 1) If $n-1 > d$, we claim that the following is a witness:

$$\mathbf{a} = [s^d, s^{d-1}, \dots, s, 1, \dots, 1]$$

Note that there is at least one 1 at the end. Thus $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$. It remains to show that $|C| \neq 0$. Note that $k = 0$ and $r = d$. Thus, the matrix C is a $(d+1) \times (d+1)$ partial block that looks like

$$C = \begin{bmatrix} & & & & 1 \\ & & & & \\ & & \ddots & & \\ & & & & \\ 1 & & & & \end{bmatrix}.$$

Therefore, $|C| = \pm 1$.

- 2) If $n-1 \leq d$ and $n-1$ divides d , we claim that the following is a witness:

$$\mathbf{a} = [s^d, s^{d-k}, \dots, s^{d-(n-1)k}]$$

Note that the last component is $s^{d-(n-1)k} = s^0 = 1$. Thus $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$. It remains to show that $|C| \neq 0$. To do this, we examine the shape of C . To get intuition, consider the instance where $n = 3$ and $d = 6$. Note that $k = 3$ and $r = 0$. Thus, we have

$$\begin{aligned} a &= [s^6, s^3, s^0] \\ C &= \begin{bmatrix} 0 & 0 & \mathbf{1} & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \mathbf{1} & 0 & 0 & 0 & 0 \\ & & & & & & & & & & \mathbf{1} \end{bmatrix} \end{aligned}$$

All the empty spaces are zeros. Note that C is a permutation matrix (each row has only one 1 and each column has only one 1). Therefore, $|C| = \pm 1$. It is easy to see that the same observation holds in general.

3) If $n - 1 \leq d$ and $n - 1$ does not divide d , we claim that the following is a witness:

$$\mathbf{a} = \left[s^d, s^{d-(1k+1)}, s^{d-(2k+2)}, \dots, s^{d-(rk+r)}, s^{d-((r+1)k+r)}, \dots, s^{d-((n-1)k+r)} \right]$$

Note that the last component is $s^{d-((n-1)k+r)} = s^0 = 1$. Thus $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$. It remains to show that $|C| \neq 0$. To do this, we examine the shape of C . To get intuition, consider the case $n = 5$ and $d = 14$. Note that $k = 3$ and $r = 2$. Thus, we have

$$a = \left[s^{14}, s^{14-(1 \cdot 3+1)}, s^{14-(2 \cdot 3+2)}, s^{14-(3 \cdot 3+2)}, s^{14-(4 \cdot 3+2)} \right] = [s^{14}, s^{10}, s^6, s^3, s^0]$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{1} & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & & & & \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & & & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & & & & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & & & & & \mathbf{1} & 0 & 0 & & & & & & & & \end{bmatrix}$$

All the empty spaces are zeros. Note that C is a permutation matrix (each row has only one 1 and each column has only one 1). Therefore, $|C| = \pm 1$. It is easy to see that the same observation holds in general.

□

We now are in a position to state the result for generic inputs.

Theorem 5. *Let \mathbb{K} be an infinite field. For generic $\mathbf{a} \in \mathbb{K}[s]^n$ with $\deg(\mathbf{a}) = d$ and $\gcd(\mathbf{a}) = 1$, for every degree-optimal moving frame P at \mathbf{a} , we have $\deg(P) = \left\lceil \frac{d}{n-1} \right\rceil$.*

Proof. From Lemma 34, it follows that $\det(C)$ is a nonzero polynomial on the $n(d+1)$ -dimensional vector space $\mathbb{K}[s]^n$ over \mathbb{K} . Thus, the condition $\det(C) \neq 0$ defines a proper

Zariski open subset of $\mathbb{K}[s]^n$. Lemma 33 implies that for every \mathbf{a} in this Zariski open subset, every degree-optimal moving frame P at \mathbf{a} has degree $\left\lceil \frac{d}{n-1} \right\rceil$. If we assume \mathbb{K} is an infinite field, then the complement of any proper Zariski open subset is of measure zero, and we can say that for a generic \mathbf{a} , the degree of every degree-optimal moving frame at \mathbf{a} equals the sharp lower bound $\left\lceil \frac{d}{n-1} \right\rceil$. \square

5 Algorithm

The theory developed in Sections 2 and 3 can be recast into an algorithm for computing a degree-optimal moving frame. After giving its informal outline, we provide an optimized version of the algorithm, trace it, and analyze its theoretical complexity.

5.1 Informal outline

Before stating a rigorous and optimized version of the algorithm, we make an informal outline and indicate how the optimization is done:

1. For an input vector $\mathbf{a} \in \mathbb{K}[s]_d^n$ such that $\gcd(\mathbf{a}) = 1$, construct the augmented matrix $W = [A \mid e_1] \in \mathbb{K}^{(2d+1) \times (nd+n+1)}$, where $A \in \mathbb{K}^{(2d+1) \times n(d+1)}$ is given by (13) and $e_1 = [1, 0, \dots, 0]^T \in \mathbb{K}^{2d+1}$.
2. Compute the reduced row-echelon form $E = [\hat{A} \mid \hat{v}]$ of W .
3. Construct a matrix $P \in \mathbb{K}[s]^{n \times n}$ whose first column is a Bézout vector of \mathbf{a} of minimal degree and whose last $n-1$ columns form a μ -basis of \mathbf{a} , as follows:
 - (a) Construct the matrix $V \in \mathbb{K}^{n(d+1) \times n}$ whose first column solves $\hat{A}v = \hat{v}$ and whose last $n-1$ columns are the null vectors of A corresponding to its basic non-pivotal columns. Here $p = [p_1, \dots, p_{2d+1}]$ is the list of the pivotal indices and $\tilde{q} = [\tilde{q}_1, \dots, \tilde{q}_{n-1}]$ is the list of the basic non-pivotal indices of A .
 - $V_{p_r,1} = \hat{v}[r]$ for $r = 1, \dots, 2d+1$
 - $V_{\tilde{q}_{j-1},j} = 1$ for $j = 2, \dots, n$
 - $V_{p_r,j} = -E_{r,\tilde{q}_{j-1}}$ for $j = 2, \dots, n$ and $r \in \{1, \dots, 2d+1 \mid p_r < \tilde{q}_{j-1}\}$
 - All other entries are zero
 - (b) Use the isomorphism \flat to convert matrix V into P :

$$P = [V_{*1}^\flat, \dots, V_{*n}^\flat].$$

However, steps 2 and 3 do some wasteful operations and they can be improved, as follows:

- Note that step 2 constructs the entire reduced row-echelon form of W , even though we only need $n-1$ null vectors corresponding to its basic non-pivot columns and the single solution vector. Hence, it is natural to optimize this step so that only the $n-1$ null vectors and the single solution vector are constructed: instead of using Gauss-Jordan elimination to compute the entire reduced row-echelon form, one performs operations column by column only on the pivot columns, basic non-pivot columns, and augmented column.

- Note that step 3 constructs the entire matrix V even though many entries are zero. Hence, it is natural to optimize this step so that we bypass constructing the matrix V , but instead construct the matrix P directly from the matrix E . This is possible because the matrix E contains all the information about the matrix V .

5.2 Formal algorithm and proof

In this section, $\text{quo}(i, j)$ denotes the quotient and $\text{rem}(i, j)$ denotes the remainder generated by dividing an integer i by an integer j .

Algorithm: OMF

Input: $\mathbf{a} \neq 0 \in \mathbb{K}[s]^n$, row vector, where $n > 1$, $\gcd(\mathbf{a}) = 1$, and \mathbb{K} a computable field

Output: $P \in \mathbb{K}[s]^{n \times n}$, a degree-optimal moving frame at \mathbf{a}

1. Construct a matrix $W \in \mathbb{K}^{(2d+1) \times (nd+n+1)}$, whose left $(2d+1) \times (nd+n)$ block is matrix (13) and whose last column is e_1 .

(a) $d \leftarrow \deg(\mathbf{a})$

(b) Identify the row vectors $c_0 = [c_{01}, \dots, c_{0n}]$, \dots , $c_d = [c_{d1}, \dots, c_{dn}]$ such that $\mathbf{a} = c_0 + c_1 s + \dots + c_d s^d$.

$$(c) \ W \leftarrow \left[\begin{array}{ccc|c} c_0 & & & 1 \\ \vdots & \ddots & & 0 \\ & & & \\ c_d & \vdots & c_0 & \vdots \\ & \ddots & \vdots & \\ & & c_d & 0 \end{array} \right] \in \mathbb{K}^{(2d+1) \times (nd+n+1)}$$

2. Construct the “partial” reduced row-echelon form E of W .

This can be done by using Gauss-Jordan elimination (forward elimination, backward elimination, and normalization), with the following optimizations:

- Skip over periodic non-pivot columns.
- Carry out the row operations only on the required columns.

3. Construct a matrix $P \in \mathbb{K}[s]^{n \times n}$ whose first column is a Bézout vector of \mathbf{a} of minimal degree and whose last $n-1$ columns form a μ -basis of \mathbf{a} .

Let p be the list of the pivotal indices and let \tilde{q} be the list of the basic non-pivotal indices of E .

(a) Initialize an $n \times n$ matrix P with 0 in every entry.

(b) For $j = 2, \dots, n$

$$r \leftarrow \text{rem}(\tilde{q}_{j-1} - 1, n) + 1$$

$$k \leftarrow \text{quo}(\tilde{q}_{j-1} - 1, n)$$

$$P_{r,j} \leftarrow P_{r,j} + s^k$$

(c) For $i = 1, \dots, 2d+1$

$$r \leftarrow \text{rem}(p_i - 1, n) + 1$$

$$\begin{aligned}
k &\leftarrow \text{quo}(p_i - 1, n) \\
P_{r,1} &\leftarrow P_{r,1} + E_{i,nd+n+1} s^k \\
\text{For } j &= 2, \dots, n \\
P_{r,j} &\leftarrow P_{r,j} - E_{i,\tilde{q}_{j-1}} s^k
\end{aligned}$$

Theorem 6. *The output of the OMF Algorithm is a degree-optimal moving frame at \mathbf{a} , where \mathbf{a} is the input vector $\mathbf{a} \in \mathbb{K}[s]^n$ such that $n > 1$ and $\gcd(\mathbf{a}) = 1$.*

Proof. In step 1, we construct a matrix $W = [A \mid e_1] \in \mathbb{K}^{(2d+1) \times (nd+n+1)}$ whose left $(2d+1) \times (nd+n)$ block is matrix (13) and whose last column is $e_1 = [1, 0, \dots, 0]^T$. Under isomorphism \flat , the null space of A corresponds to $\text{syz}_d(\mathbf{a})$, and the solutions to $Av = [1, 0, \dots, 0]^T$ correspond to $\text{Bez}_d(\mathbf{a})$. From Lemma 28, we know that $\text{rank}(A) = 2d+1$, and thus all pivotal columns of W are the pivotal columns of A . In step 2, we perform partial Gauss-Jordan operations on W to identify the coefficients α 's appearing in (25) and (19), that express the $n-1$ basic non-pivotal columns of A and the vector e_1 , respectively, as linear combinations of pivotal columns of A . These coefficients will appear in the basic non-pivotal columns and the last column of the partial reduced row-echelon form E of W . In Step 3, we use these coefficients to construct a minimal-degree Bézout vector of \mathbf{a} and a degree-ordered μ -basis of \mathbf{a} , as prescribed by Theorems 3 and 4. We place these vectors as the columns of matrix P , and the resulting matrix is, indeed, a degree-optimal moving frame according to Theorem 1. \square

5.3 Example tracing the algorithm

Example 35. *We trace the algorithm on the input vector*

$$\mathbf{a} = \begin{bmatrix} 2 + s + s^4 & 3 + s^2 + s^4 & 6 + 2s^3 + s^4 \end{bmatrix} \in \mathbb{Q}[s]^3$$

1. Construct matrix W :

(a) $d \leftarrow 4$

(b) $c_0, c_1, c_2, c_3, c_4 \leftarrow \begin{bmatrix} 2 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$$(c) \ W \leftarrow \left[\begin{array}{ccc|ccc|ccc|ccc|ccc|c}
2 & 3 & 6 & & & & & & & & & & & & 1 \\
1 & 0 & 0 & 2 & 3 & 6 & & & & & & & & & \\
0 & 1 & 0 & 1 & 0 & 0 & 2 & 3 & 6 & & & & & & \\
0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 3 & 6 & & & \\
1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 3 & 6 \\
& & & 1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\
& & & & & & 1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\
& & & & & & & & & 1 & 1 & 1 & 0 & 0 & 2 \\
& & & & & & & & & & & & 1 & 1 & 1
\end{array} \right]$$

2. Construct the “partial” reduced row-echelon form E of W .

$$E \leftarrow \left[\begin{array}{ccc|ccc|ccc|c} 1 & & & & & & & & & 2 \\ & 1 & & & & & & & & 1 \\ & & 1 & & & & & & & -1 \\ & & & 1 & & & & & & -1 \\ & & & & 1 & & & & & 2 \\ & & & & & 1 & & & & -1 \\ & & & & & & 1 & & & 0 \\ & & & & & & & 1 & & 0 \\ & & & & & & & & 1 & 0 \\ & & & & & & & & & 0 \\ & & & & & & & & & 0 \\ & & & & & & & & & 0 \end{array} \right]$$

Here, blue denotes pivotal columns, red denotes basic non-pivotal columns, brown denotes periodic non-pivotal columns, and grey denotes the solution column.

3. Construct a matrix $P \in \mathbb{K}[s]^{n \times n}$ whose first column consists of a minimal-degree Bézout vector of \mathbf{a} and whose last $n - 1$ columns form a μ -basis of \mathbf{a} .

$$(a) \ P \leftarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(b) \ P \leftarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s^2 \end{bmatrix}$$

$$(c) \ P \leftarrow \begin{bmatrix} 2 - s & 3 - 3s - s^2 & 9 - 12s - s^2 \\ 1 + 2s & 2 + 5s + s^2 & 8 + 15s \\ -1 - s & -2 - 2s & -7 - 5s + s^2 \end{bmatrix}$$

5.4 Theoretical Complexity Analysis

In this subsection, we analyze the theoretical (asymptotic worst case) complexity of the OMF algorithm given in Section 5.2. We will do so under the assumption that the time for any arithmetic operation is constant.

Proposition 36. *The complexity of the OMF algorithm is*

$$O(d^2n + d^3 + n^2).$$

Proof. We will trace the theoretical complexity for each step of the algorithm.

1. (a) To determine d , we scan through each of the n polynomials in \mathbf{a} to identify the highest degree term, which is always $\leq d$. Thus, the complexity for this step is $O(dn)$.

- (b) We identify $n(d+1)$ values to make up c_0, \dots, c_d . Thus, the complexity for this step is $O(dn)$.
- (c) We construct a matrix with $(2d+1)(nd+n+1)$ entries. Thus, the complexity for this step is $O(d^2n)$.
- 2. With the partial Gauss-Jordan elimination, we perform row operations only on the $2d+1$ pivot columns of A , the $n-1$ basic non-pivot columns of A , and the augmented column e_1 . Thus, we perform Gauss-Jordan elimination on a $(2d+1) \times (2d+n+1)$ matrix. In general, for a $k \times l$ matrix, Gauss-Jordan elimination has complexity $O(k^2l)$. Thus, the complexity for this step is $O(d^2(d+n))$.
- 3. (a) We fill 0 into the entries of an $n \times n$ matrix P . Thus, the complexity for this step is $O(n^2)$.
- (b) We update entries of the matrix $n-1$ times. Thus, the complexity for this step is $O(n)$.
- (c) We update entries of the matrix $(2d+1)(n-1)$ times. Thus, the complexity for this step is $O(dn)$.

By summing up, we have

$$O(dn + dn + d^2n + d^2(d+n) + n^2 + n + dn) = O(d^2n + d^3 + n^2)$$

□

Remark 37. Note that the n^2 term in the above complexity is solely due to step 3(a), where the matrix P is initialized with zeros. If one uses a sparse representation of the matrix (storing only nonzero elements), then one can skip the initialization of the matrix P . As a result, the complexity can be improved to $O(d^2n + d^3)$.

It turns out that the theoretical complexity of the OMF algorithm is exactly the same as that of the μ -basis algorithm presented in [10]. This is unsurprising, because the μ -basis algorithm presented in [10] is based on partial Gauss-Jordan elimination of matrix A , while the OMF algorithm is based on partial Gauss-Jordan elimination of the matrix obtained by appending to A a single column e_1 .

6 Discussion

In this section, we discuss other algorithms for constructing moving frames satisfying Definition 3 and make a comparison between these moving frames and moving frames appearing in differential geometry.

6.1 Comparison with other moving frame algorithms

In Section 2, we outlined an informal algorithm for producing a degree-optimal moving frame at \mathbf{a} . Namely, construct a minimal-degree Bézout vector of \mathbf{a} , construct a μ -basis of \mathbf{a} , and then combine them in a matrix. However, we are not aware of any algorithms for constructing a minimal-degree Bézout vector. Using the ideas presented

in the proof of Theorem 2, it is possible to reduce the degree of a Bézout vector of \mathbf{a} using a μ -basis of \mathbf{a} . However, such a process (construct Bézout vector, construct μ -basis, reduce) is inefficient, and it still does not guarantee that the Bézout vector after reduction will be minimal-degree. The advantage of the OMF algorithm is that not only does it construct a minimal-degree Bézout vector of \mathbf{a} and a μ -basis of \mathbf{a} , but it does so simultaneously and with just a single “partial” Gauss-Jordan elimination.

A simple and elegant algorithm for constructing not-necessarily-optimal moving frames, based on a generalized version of Euclid’s extended gcd algorithm, can be extracted from Theorem B.1.16 of the book “Introduction to the Mathematical Theory of Systems and Control” by Polderman and Willems [12]. For the readers’ convenience, we describe this algorithm in our notation (as a recursive program for simple presentation). We will refer to this algorithm as MF_GE (abbreviation of “Moving Frame by Generalized Euclid’s algorithm”).

Input: $\mathbf{a} \in \mathbb{K}[s]^n$, $\mathbf{a} \neq 0$

Output: P , a moving frame of \mathbf{a}

1. Let k be such that $\mathbf{a} = [a_1 \ \cdots \ a_k \ 0 \ \cdots \ 0]$ where $a_k \neq 0$.
2. If $k = 1$ then set

$$P = \begin{bmatrix} \frac{1}{\text{lc}(a_1)} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

and return P . (Here, $\text{lc}(a_1)$ denotes the leading coefficient of a_1 .)

3. (Find $q_2, \dots, q_k, r \in \mathbb{K}[s]^n$ such that $a_1 = q_2 a_2 + \dots + q_k a_k + r$.)
 - (a) $r \leftarrow a_1$
 - (b) For $i = 2, \dots, k$ do
 - $q_i \leftarrow \text{quo}(r, a_i)$
 - $r \leftarrow \text{rem}(r, a_i)$

4. $\mathbf{a}' \leftarrow [a_2 \ \cdots \ a_k \ r \ 0 \ \cdots \ 0]$.

$$5. \ T \leftarrow \begin{bmatrix} & & 1 & & \\ & 1 & -q_2 & & \\ & & \vdots & & \\ & & & 1 & -q_k \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \in \mathbb{K}[s]^{n \times n},$$

where the q ’s are placed in the k -th column

6. $P' \leftarrow \text{MF_GE}(\mathbf{a}')$.
7. $P \leftarrow TP'$.
8. Return P .

Remark 38. We make a few remarks on the MF-GE algorithm.

- The MF-GE algorithm is elegant and very simple. The correctness is immediate from the fact that P is a product of uni-modular matrices (up to a sign).
- The MF-GE algorithm works even when $\gcd(\mathbf{a}) \neq 1$, unlike our algorithm OMF.
- For $n = 2$, the MF-GE algorithm always produces a degree-optimal moving frame. This is unsurprising, since when $n = 2$, the algorithm returns a matrix whose first column is the output of the standard extended Euclidean algorithm which, for two polynomials, is known to produce a minimal-degree Bézout vector. The second column of the output is the obvious lowest degree syzygy $\frac{1}{\gcd(\mathbf{a})}[-a_2, a_1]$.
- For $n > 2$, the MF-GE algorithm does not always produce a degree-optimal moving frame. For instance, for our running example (Example 5), the MF-GE output is

$$\begin{bmatrix} \frac{1}{2} + \frac{1}{2}s + \frac{1}{2}s^2 & -\frac{3}{2}s + \frac{1}{2}s^2 - \frac{1}{2}s^3 & \frac{3}{2} - \frac{3}{2}s - \frac{1}{2}s^2 \\ -\frac{1}{2}s - \frac{1}{2}s^2 & 2 + s - \frac{1}{2}s^2 + \frac{1}{2}s^3 & 1 + \frac{5}{2}s + \frac{1}{2}s^2 \\ 0 & -1 & -1 - s \end{bmatrix}$$

and the OMF output is

$$\begin{bmatrix} 2 - s & 3 - 3s - s^2 & 9 - 12s - s^2 \\ 1 + 2s & 2 + 5s + s^2 & 8 + 15s \\ -1 - s & -2 - 2s & -7 - 5s + s^2 \end{bmatrix}.$$

Observe that the degree of the Bézout vector column (i.e. the first column) for MF-GE is 2, while the degree of the Bézout vector column for OMF is 1. Likewise, the degrees of the syzygy columns for MF-GE are 2 and 3, while the degrees of the syzygy columns for OMF are 2 and 2.

An algorithm very similar to the MF-GE was used, as an auxiliary computation, in [5] by Elkadi, Galligo, and Ba. Their paper is devoted to the following problem: given a vector of polynomials, find small degree perturbations so that the perturbed polynomials have a large-degree gcd. As discussed in Example 3 of [5], the perturbations produced by the algorithm presented in this paper do not always have minimal degrees. It would be interesting to study if the usage of degree-optimal moving frames would decrease the degrees of the perturbations. This investigation lies outside of the scope of the current paper. We also would like to explore possible applications for our OMF algorithm to the theory of systems and control in the setting of [12].

6.2 Comparison with geometric moving frames

The notion of a moving frame can be traced back to the work of Euler [6], who noticed that the description of the motion of a rigid body can be simplified by utilizing a frame of reference that is intrinsic to the body, instead of using a fixed external system of coordinates. In differential geometry, a classical example of a moving frame is provided by the Frenet-Serret frame for a curve in \mathbb{R}^3 consisting of the unit tangent, normal, and binormal vectors. The Frenet-Serret frame is equivariant with respect to the action of

the Euclidean group (rigid motions) in the following sense. Let γ be a curve in \mathbb{R}^3 and let g be an element of the Euclidean group. If $[T, N, B]$ is the Frenet-Serret frame at a point p of the curve γ , then $[gT, gN, gB]$ is the Frenet-Serret frame at a point gp of the curve $g\gamma$. The group-equivariance property is essential for the majority of frames arising in differential geometry, notably in the works of Cartan, such as [4], who used moving frames to compute differential invariants and to solve various group-equivalence problems in geometry. Building on the subsequent works of Griffiths [9] and Green [8], in [7], Fels and Olver gave a very general definition of a group-equivariant moving frame, which can be outlined as follows. Let a group G act on a manifold M . Then a moving frame is a smooth equivariant map $\rho: M \rightarrow G$. A moving frame exists if the action is free and regular. If a moving frame exists, then for every point $p \in M$, the point $\rho(p)p$ provides a canonical representative of the orbit of p . Group actions of interest are often not free, but they can be prolonged to a free action on a larger space; for instance, to a jet space of submanifolds of M . Alternatively, instead of constructing an equivariant map $\rho: M \rightarrow G$, one can construct, under certain conditions (see for instance [11]), an equivariant map ρ from M to a space G/H , where H is some subgroup of G , in such a way that any representative $[\rho(p)]$ of the coset $\rho(p)$ still maps a point p to a canonical representative of the orbit of p .

We can put our construction in this latter context by considering an action of the group $GL_n(\mathbb{K}[s])$ on the set of polynomial vectors $\mathbb{K}[s]^n$. Then, it is not difficult to show that the orbit of an arbitrary nonzero polynomial vector \mathbf{a} contains a unique representative of the form $[f, 0, \dots, 0]$, where f is a monic polynomial. Moreover, one can show that $f = \gcd(\mathbf{a})$. One also can show that the stabilizer subgroup of any canonical representative $[f, 0, \dots, 0]$ is the affine group $A_{n-1}(\mathbb{K}[s])$ comprised of polynomial $n \times n$ matrices of the form:

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{h} & \mathbf{M} \end{bmatrix},$$

where the column vectors \mathbf{h} and the row vector $\mathbf{0}$ are elements of $\mathbb{K}[s]^{n-1}$, while $\mathbf{M} \in GL_{n-1}(\mathbb{K}[s])$ is a polynomial matrix with constant determinant. One can easily see that any algorithm for constructing a moving frame $P(\mathbf{a})$ at a nonzero input vector \mathbf{a} defines an equivariant map ρ from the set of nonzero polynomial vectors $\mathbb{K}[s]^n \setminus \{\mathbf{0}\}$ to the homogeneous space $GL_n(\mathbb{K}[s]) \setminus A_{n-1}(\mathbb{K}[s])$, with $P(\mathbf{a})$ serving as a representative of the coset $\rho(\mathbf{a})$. By definition, $P(\mathbf{a})$ brings \mathbf{a} to the canonical representative of its orbit, $[\gcd(\mathbf{a}), 0, \dots, 0]$.

In the introduction, we justified the term moving frame by picturing it as a coordinate system moving along a curve. This point of view is reminiscent of classical geometric frames, such as the Frenet-Serret frame. In this context, the relevant group would be $GL_n(\mathbb{K})$, the group of invertible matrices with *constant* entries, if we work with polynomial curves in \mathbb{K}^n , or the projective group $PGL_n(\mathbb{K})$ if we work in the projective space $\mathbb{P}\mathbb{K}^{n-1}$. The following proposition shows that the degree optimality property is $GL_n(\mathbb{K})$ -equivariant.

Proposition 39. *Let P be a degree-optimal moving frame at a nonzero polynomial vector \mathbf{a} . Then, for any $g \in GL_n(\mathbb{K})$, the matrix $g^{-1}P$ is a degree-optimal moving frame at the vector $\mathbf{a}g$.*

Proof. By definition, $\mathbf{a}P = [\gcd(\mathbf{a}), 0, \dots, 0]$ and, therefore, for any $g \in GL_n(\mathbb{K})$ we have:

$$(\mathbf{a}g)g^{-1}P = [\gcd(\mathbf{a}), 0, \dots, 0].$$

From this, we conclude that $\gcd(\mathbf{a}g) = \gcd(\mathbf{a})$ and that $g^{-1}P$ is a moving frame at $\mathbf{a}g$. We note that the rows of the matrix $g^{-1}P$ are linear combinations over \mathbb{K} of the rows of the matrix P . Therefore, the degrees of the columns of $g^{-1}P$ are less than or equal to the degrees of the corresponding columns of P .

Assume that $g^{-1}P$ is not a degree-optimal moving frame at $\mathbf{a}g$. Then there exists a moving frame P' at $\mathbf{a}g$ such that at least one of the columns of P' , say the j -th column, has degree strictly less than the j -th column of $g^{-1}P$. Then, from the paragraph above, the j -th column of P' has degree strictly less than the degree of the j -th column of P .

By the same argument, gP' is a moving frame at \mathbf{a} such that its j -th column has degree less than or equal to the degree of the j -th column of P' , which is strictly less than the degree of the j -th column of P . This contradicts our assumption that P is degree-optimal. \square

Although the degree optimality property is $GL_n(\mathbb{K})$ -equivariant, it is not a trivial task to design a $GL_n(\mathbb{K})$ -equivariant algorithm for producing a degree-optimal moving frame. The OMF algorithm *is not* $GL_n(\mathbb{K})$ -equivariant: in general, for $\mathbf{a} \in \mathbb{K}[s]^n$ and $g \in GL_n(\mathbb{K})$,

$$OMF(\mathbf{a}g) \neq g^{-1}OMF(\mathbf{a}).$$

Since the group-equivariance property is important for many applications, designing a $GL_n(\mathbb{K})$ -equivariant degree-optimal moving frame algorithm is an interesting topic for future research.

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